[A large fraction of the contents are adapted from Prof. Kunz's Lecture Notes on Introduction to Plasma Astrophysics, Part V]

MHD Waves

MHD Equations.

continuity. $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$

momentum. $\rho(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}) = \frac{\vec{J}\times\vec{B}}{c} - \vec{\nabla}p$

– Ohm's law: $\vec{E} + \frac{\vec{u}\times\vec{B}}{c} = 0$

Ampere's law. $\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B}$

Faraday's law. $\frac{\partial\vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$

Induction Eqn. $\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$

Adiabatic energy Eqn. $(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla)(\frac{p}{\rho^\gamma}) = 0$

γ: ratio of specific heat.

Recall that Prof. Kunz explained the meaning of each term in the $\vec{J}\times\vec{B}$ force and in $\vec{\nabla}\times(\vec{u}\times\vec{B})$ term

understand how $\vec{B}$ acts on $\vec{u}$

understand how $\vec{u}$ acts on $\vec{B}$

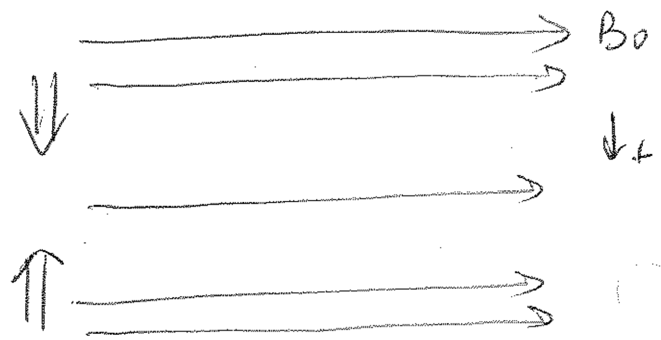
$$F_M = \frac{\vec{J}\times\vec{B}}{c} = \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi} - \vec{\nabla}\frac{B^2}{8\pi} = -\vec{\nabla}\cdot\Big[\underbrace{\frac{B^2}{8\pi}\overleftrightarrow{I}}_{B\text{ pressure}} - \frac{\vec{B}\vec{B}}{4\pi}\Big] \equiv -\vec{\nabla}\cdot\underbrace{\overleftrightarrow{M}}_{\text{Maxwell stress}}$$

~~off~~ ~~dig~~ diaognal terms, B. introduces a special direction. / anisotropy.

Def. $\hat{b} \equiv \frac{\vec{B}}{B}$.

$$F_M = \frac{\vec{J}\times\vec{B}}{c} = \frac{B^2}{4\pi}\underbrace{\hat{b}\cdot\nabla\hat{b}}_{\frac{1}{r_c}} - \underbrace{(\vec{I}-\hat{b}\hat{b})\cdot\vec{\nabla}}_{\vec{\nabla}_\perp}\frac{B^2}{8\pi}$$

curvature force pressure force



B field has the tendency to be straight and spread out.

→ The plasma gains elasticity, distinct from hydro fluid.

Linear theory

Equilibrium. $\vec{B}_0$, ρ_0, P_0, $\vec{u}_0 = 0$ (steady / static) assume

now we perturb the system,
Introducing a small deviation for each quantity.

$\vec{B} = \vec{B}_0 + \vec{B}_1$ $\quad \rho = \rho_0 + \rho_1$

$P = P_0 + P_1$ $\quad \vec{u} = \vec{u}_0 + \vec{u}_1 = \vec{u}_1$

$$\begin{cases} \frac{\partial Q_0}{\partial t} = 0 \\ \nabla Q_0 = 0 \end{cases}$$

Key for applying linear theory:
the perturbation is small, so that

$$\frac{B_1}{B_0} \sim \frac{\rho_1}{\rho_0} \sim \frac{P_1}{P_0} \sim O(\varepsilon), \quad \varepsilon \text{ small}$$

Linearizing MHD Eqns:

→ density: $\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla \cdot [(\rho_0 + \rho_1)\vec{u}_1] = 0$

$O(1)$: $\frac{\partial}{\partial t}\rho_0 = 0$.

$O(\varepsilon)$: $\frac{\partial}{\partial t}\rho_1 + \nabla \cdot \rho_0 \vec{u}_1 = 0$, neglecting $\nabla \cdot \rho_1 \vec{u}_1$ $O(\varepsilon^2)$

perturbation is small enough that
the nonlinear ~~of~~ perturbeds terms can be neglected.

→ momentum: $(\rho_0 + \rho_1)\left[\frac{\partial}{\partial t} + \vec{u}_1 \cdot \nabla\right]\vec{u}_1 = -\nabla(P_0 + P_1) - \frac{\nabla(\vec{B}_0 + \vec{B}_1)^2}{8\pi} \cancel{+ \frac{1}{4\pi}\vec{B}_0 \cdot \nabla \vec{B}_1}$
$+ \frac{1}{4\pi}(\vec{B}_0 + \vec{B}_1) \cdot \nabla(\vec{B}_0 + \vec{B}_1)$

$O(1)$: $0 = -\nabla P_0 - \nabla \frac{B_0^2}{8\pi} = 0$

$O(\varepsilon)$: $\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\nabla P_1 - \frac{1}{4\pi}\nabla(\vec{B}_0 \cdot \vec{B}_1) + \frac{1}{4\pi}\vec{B}_0 \cdot \nabla \vec{B}_1$

$\rightarrow$ Induction eqn.

$$\frac{\partial(\vec{B}_0+\vec{B}_1)}{\partial t} = \nabla\times[\vec{u}_1\times(\vec{B}_0+\vec{B}_1)]$$

$O(1)$: $\quad \frac{\partial \vec{B}_0}{\partial t} = 0$

$O(\varepsilon)$: $\quad \frac{\partial \vec{B}_1}{\partial t} = \nabla\times(\vec{u}_1\times\vec{B}_0)$

$$= (\vec{B}_0\cdot\nabla)\vec{u}_1 - \underbrace{\cancel{(\vec{u}_1\cdot\nabla)\vec{B}_0}}_{\text{uniform } B_0} - \vec{B}_0(\nabla\cdot\vec{u}_1) + \underbrace{\cancel{\vec{u}_1(\nabla\cdot\vec{B}_0)}}_{\nabla\cdot B=0}$$

$\rightarrow$ energy eqn.

$$\left(\frac{\partial}{\partial t}+\cancel{\vec{u}_1\cdot\nabla}\right)\left(\frac{P_0+P_1}{(\rho_0+\rho_1)^\gamma}\right)=0$$

$$(P_0+P_1)(\rho_0+\rho_1)^{-\gamma} = (P_0+P_1)\rho_0^{-\gamma}\left(1+\frac{\rho_1}{\rho_0}\right)^{-\gamma}$$

keep $O(\varepsilon)$ for taylor expansion.

$$\simeq (P_0+P_1)\rho_0^{-\gamma}\left(1-\gamma\frac{\rho_1}{\rho_0}\right)$$

$$= \frac{P_0}{\rho_0^\gamma}\left(1+\frac{P_1}{P_0}\right)\left(1-\gamma\frac{\rho_1}{\rho_0}\right)$$

$O(1)$: $\quad \frac{\partial}{\partial t}\frac{P_0}{\rho_0^\gamma}=0$

$O(\varepsilon)$: $\quad \frac{P_0}{\rho_0^\gamma}\frac{\partial}{\partial t}\left[\frac{P_1}{P_0}-\gamma\frac{\rho_1}{\rho_0}\right]=0 \quad \Rightarrow \quad \frac{P_1}{P_0}=\gamma\frac{\rho_1}{\rho_0}$

Linearized MHD Eqns (for static, uniform background)

$$\begin{cases} \frac{\partial}{\partial t}\rho_1 + \nabla\cdot\rho_0\vec{u}_1 = 0 \\ \rho_0\frac{\partial\vec{u}_1}{\partial t} = -\nabla P_1 - \frac{1}{4\pi}\nabla(\vec{B}_0\cdot\vec{B}_1) + \frac{1}{4\pi}\vec{B}_0\cdot\nabla\vec{B}_1 \\ \frac{\partial\vec{B}_1}{\partial t} = (\vec{B}_0\cdot\nabla)\vec{u}_1 - \vec{B}_0(\nabla\cdot\vec{u}_1) \\ \frac{P_1}{P_0} = \gamma\frac{\rho_1}{\rho_0} \end{cases}$$

Adopt plane-wave solution for ~~perturbed~~ perturbation.

$$Q_1 \xrightarrow{\text{Fourier Transform}} \hat{Q}_1 \exp(-i\omega t + i\vec{k}\cdot\vec{r}).$$

$$\frac{\partial}{\partial t} \to -i\omega \quad , \quad \vec{\nabla} \to i\vec{k}$$

Now perform Fourier Transform to linearized eqns.

$\cancel{-i\omega\rho_1 + i\vec{k}\cdot\vec{u}_1\rho_0 = 0}$

$\cancel{-\rho_0 i\omega\vec{u}_1 = -i\vec{k}p_1 - \frac{1}{4\pi}}$

$\vec{k} = \vec{k}_\parallel + \vec{k}_\perp$

assume $\vec{k}_\parallel$, $\vec{B}_0$ in $\hat{z}$

$$\begin{cases} -i\omega \dfrac{\rho_1}{\rho_0} + i\vec{k}\cdot\vec{u} = 0 \\ -i\omega \vec{u}_1 = -\dfrac{i\vec{k}}{\rho_0}\left(p_1 + \dfrac{B_0 B_{1\parallel}}{4\pi}\right) + \dfrac{ik_\parallel}{4\pi\rho_0} B_0 \vec{B}_1 \\ -i\omega \dfrac{\vec{B}_1}{B_0} = ik_\parallel \vec{u}_1 - \hat{z}\, i\vec{k}\cdot\vec{u} \\ \dfrac{p_1}{p_0} = \gamma \dfrac{\rho_1}{\rho_0} \end{cases}$$

"^" is dropped.

Let's first consider some special cases.

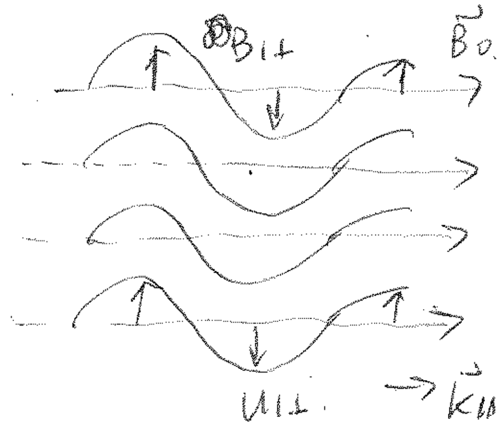
1). ~~Assume~~ Shear Alfven wave.

assume $\vec{k} = \vec{k}_\parallel$

$B_{1\parallel} = 0$, no compression in B

$\rho_1 = 0$., no density pert.

treat magnetic fields like string.



$$\begin{cases} \rho_1 = 0. \\ -i\omega \vec{u}_{1\perp} = + \dfrac{i k_\parallel}{\rho_0 4\pi} B_0 \vec{B}_{1\perp} \\ -i\omega \dfrac{\vec{B}_{1\perp}}{B_0} = i k_\parallel \vec{u}_{1\perp} \end{cases}$$

get rid of $u_{1\perp}$:
$$\frac{k_\parallel}{4\pi\rho_0\omega} B_0 \vec{B}_{1\perp} = \omega \frac{B_{1\perp}}{B_0} \frac{1}{k_\parallel}$$

$$\left[\omega^2 - k_\parallel^2 \frac{B_0^2}{4\pi\rho_0}\right] \frac{B_{1\perp}}{B_0} = 0$$

dispersion relation: $\omega^2 = k_\parallel^2 V_A^2$. $V_A \equiv \dfrac{B_0}{\sqrt{4\pi\rho_0}}$

Alfvén speed.

pluck a string (B field line),
tension force pulls the string back.

"Shear Alfvén wave"

1970 Nobel Prize.

Alfven waves propogate forward and backward can interact nonlinearly, transfer energy to smaller scales

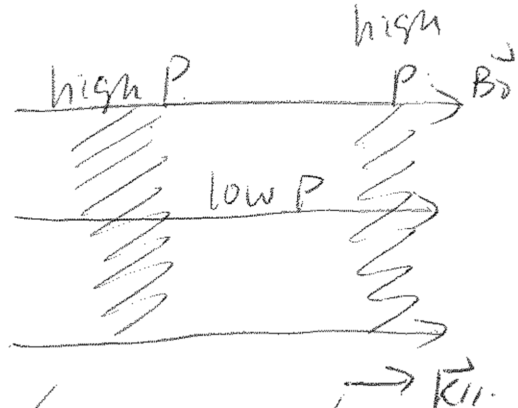
$\Rightarrow$ building block of magnetized turbulence.

2). Sound wave

assume. $\vec{k} = \vec{k}_{\parallel}$

$\vec{B}_{1\perp} = 0$ (no field line bending).

$\vec{u}_{1\perp} = 0$



$$\begin{cases} -i\omega \frac{\rho_1}{\rho_0} + ik_{\parallel} u_{1\parallel} = 0 \\ -i\omega u_{1\parallel} = -\frac{ik_{\parallel}}{\rho_0}\left(P_1 + \cancel{\frac{B_0 B_{1\parallel}}{4\pi}}\right) + \cancel{\frac{ik_{\parallel}}{4\pi\rho_0} B_0 B_{1\parallel}} \\ \frac{P_1}{P_0} = \gamma \frac{\rho_1}{\rho_0} \end{cases}$$

no compression on B.

eliminate $u_{1\parallel}$, $\frac{\rho_1}{\rho_0} + \frac{k_{\parallel}}{\omega^2}\frac{k_{\parallel}}{\rho_0} P_1 = 0$.

$$\left[\omega^2 - \frac{\gamma P_0}{\rho_0} k_{\parallel}^2\right] \frac{\rho_1}{\rho_0} = 0$$

$\omega^2 = c_s^2 k_{\parallel}^2$. $c_s \equiv \sqrt{\frac{\gamma P_0}{\rho_0}}$, sound speed.

$\boxed{\text{sound wave}}$

propogation of pressure perturbation.

Shear Alfvén wave and sound wave are two basic elements.

a combination of the two results in a variety of waves.

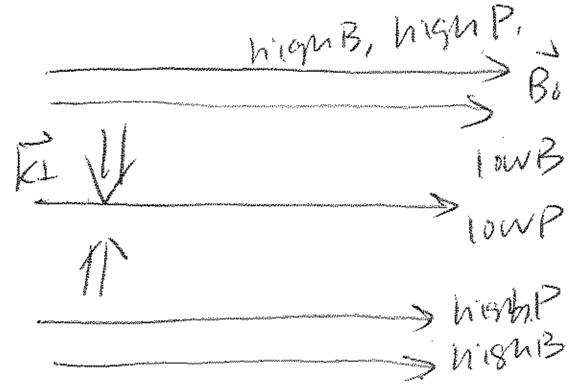
3). Magneto-sonic wave.

a special case, B pressure + gas pressure work together.

Assume.

$\vec{k} = \vec{k}_\perp$ (no B line bending).

$\vec{B}_{1\perp} = 0$



$$\left\{\begin{array}{ll} -i\omega \frac{\rho_1}{\rho_0} + i k_\perp u_\perp = 0. & -① \\ -i\omega u_{1\perp} = -i\frac{k_\perp}{\rho_0}\left(P_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right) & -② \\ -i\omega \frac{B_{1\parallel}}{B_0} = -i k_\perp u_\perp & -③ \\ \frac{P_1}{P_0} = \gamma \frac{\rho_1}{\rho_0} & \end{array}\right.$$

plug ① ④ into ②, ③, eliminate $\rho_1, u_{1\perp}$.

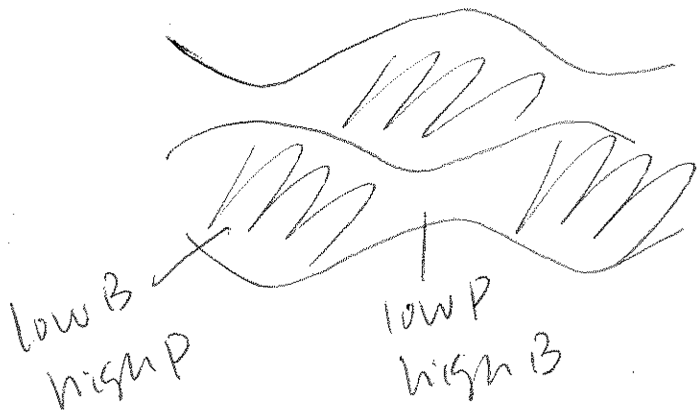
$$\omega^2 \frac{1}{\gamma}\frac{P_1}{P_0}\frac{1}{k_\perp} = \frac{k_\perp}{\rho_0} P_1 + \frac{k_\perp}{\rho_0}\frac{B_0 B_{1\parallel}}{4\pi}$$

$$\cancel{\omega}\frac{B_{1\parallel}}{B_0} = \frac{\cancel{\omega}}{\gamma}\frac{P_1}{P_0}$$

$$\frac{B_{1\parallel}}{B_0}\left[\frac{\omega^2}{k_\perp^2} - \frac{\gamma P_0}{\rho_0} - \frac{B_0^2}{4\pi\rho_0}\right] = 0$$

$$\omega^2 = k_\perp^2 \underbrace{(C_s^2 + V_A^2)}_{\text{B, gas pressure acting together.}}$$

~~In~~ On top of that, allow field line bending ($k_{\parallel} \neq 0$).



anti-correlated ρ_1, B_1.
~~Mea~~ P_1, B pressure out of phase.

slow mode: launching wave slower

$$\omega^2 = k_{\parallel}^2 V_A^2 \cdot \left(\frac{C_s^2}{C_s^2 + V_A^2} \right)$$

Now we can consider a general case.

$\vec{k} = k_{\parallel}\hat{z} + \vec{k_\perp}$, keep $\rho_1, \vec{B_{1\perp}}, \vec{B_{1\parallel}}, \vec{u_{1\perp}}, \vec{u_{1\parallel}}, P_1$.

(a) $i\omega \frac{\rho_1}{\rho_0} + i k_{\parallel} u_{1\parallel} + i \vec{k_\perp} \cdot \vec{u_\perp} = 0$

(d) $-i\omega \vec{u_{1\perp}} = \frac{i\vec{k_\perp}}{\rho_0}\left(P_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right) + \frac{i k_{\parallel} B_0}{4\pi\rho} \vec{B_{1\perp}}$

(e) $i\omega \vec{u_{1\parallel}} = \frac{-i k_{\parallel}}{\rho_0}\left(P_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right) + \frac{i k_{\parallel} B_0}{4\pi\rho} \vec{B_{1\parallel}}$

(b) $-i\omega \frac{\vec{B_{1\perp}}}{B_0} = i k_{\parallel} \vec{u_{1\perp}}$

(c) $-i\omega \frac{\vec{B_{1\parallel}}}{B_0} = -i \vec{k_\perp} \cdot \vec{u_{1\perp}}$

(d)$\cdot \vec{k_\perp} + k_{\parallel}$(e) : $-i\omega(\vec{k_\perp} \cdot \vec{u_{1\perp}} + k_{\parallel} u_{1\parallel}) = -i \frac{k^2}{\rho_0}\left(P_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right)$

plug in (a) $\omega \frac{\rho_1}{\rho_0} = \frac{k^2}{\rho_0}\left(P_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right)$

using $\frac{p_1}{p_0} = \gamma \frac{\rho_1}{\rho_0}$: $(\omega^2 - k^2 c_s^2)\frac{\rho_1}{\rho_0} = k^2 V_A^2 \frac{B_{1\parallel}}{B_0}$

using (d) and (b) gives

$$(\omega^2 - k_\parallel^2 V_A^2)\frac{\vec{B}_{1\perp}}{B_0} = -k_\parallel \vec{k}_\perp \left(c_s^2 \frac{\rho_1}{\rho_0} + V_A^2 \frac{B_{1\parallel}}{B_0}\right)$$

plugs in $\frac{p_1}{p_0} = \beta\gamma \frac{\rho_1}{\rho_0}$

we get $(\omega^2 - k_\parallel^2 V_A)\frac{\vec{B}_{1\perp}}{B_0} = -k_\parallel \vec{k}_\perp \frac{B_{1\parallel}}{B_0}\left[c_s^2 \frac{k^2 V_A^2}{\omega^2 - k^2 c_s^2} + V_A^2\right]$

$$= -k_\parallel \vec{k}_\perp V_A^2 \left[\frac{\omega^2}{\omega^2 - k^2 c_s^2}\right]\frac{B_{1\parallel}}{B_0}$$

(b) and (c) gives $\frac{B_{1\parallel}}{B_0} = -\frac{\vec{k}_\perp \cdot \vec{B}_{1\perp}}{k_\parallel B_0}$,

we get

$$\left[\overleftrightarrow{I}(\omega^2 - k_\parallel^2 V_A^2) - \vec{k}_\perp \vec{k}_\perp V_A^2 \frac{c_s^2}{\omega^2 - k^2 c_s^2}\right] \cdot \frac{\vec{B}_{1\perp}}{B_0} = 0$$

setting the determinant to zero:

$$(\omega^2 - k_\parallel^2 V_A^2)\left[\omega^4 - \omega^2 k^2 (c_s^2 + V_A^2) + k_\parallel^2 V_A^2 k^2 c_s^2\right] = 0$$

recover shear Alfven wave $\underline{\omega = \pm k_\parallel V_A}$

Also have:

$$\omega^2 = \frac{k^2 (c_s^2 + V_A^2)}{2} \pm \sqrt{\frac{k^4 (c_s^2 + V_A^2)^2}{4} - k_\parallel^2 V_A^2 k^2 c_s^2}$$

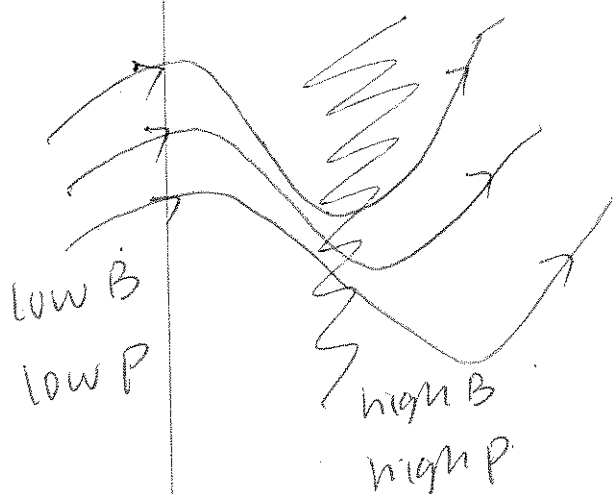
magnetosonic mode: $+$: fast mode; $-$: slow mode.

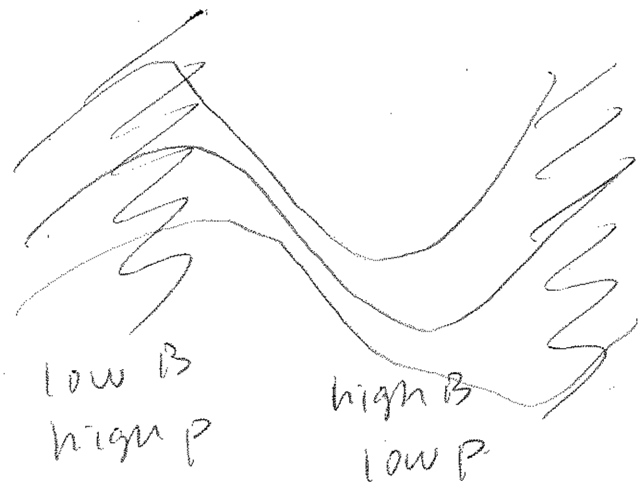
in limit $\frac{k_\parallel}{k_\perp} \ll 1$:

$$\omega^2 \simeq \frac{k_\perp^2 (c_s^2 + V_A^2)}{2}\left[1 \pm \left(1 - \frac{2 k_\parallel^2 V_A^2 k_\perp^2 c_s^2}{k_\perp^4 (c_s^2 + V_A^2)}\right)\right]$$

Fast

Slow





magnetic field pressure
&
gas pressure
in phase

out of phase