NSF/GPAP SUMMER SCHOOL

on plasma physics for astrophysicists

Nonlinear Evolution of MHD Instabilities

Jim Stone

Institute for Advanced Study

Motivation: the complex structure of astrophysical plasmas

Carina nebula, HST image

Start with the equations of compressible viscous and resistive MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + \mathsf{P}^* + \Pi] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}\,(\mathbf{B} \cdot \mathbf{v}) + \Pi \cdot \mathbf{v} + \eta \mathbf{J} \times \mathbf{B}] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \mathbf{J} = 0$$

$$P^* = P + \frac{\mathbf{B} \cdot \mathbf{B}}{2}$$

$$E = P/(\gamma - 1) + \frac{\rho(\mathbf{v} \cdot \mathbf{v})}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}$$

$$\Pi_{ij} = \rho \nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_j} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right)$$

Also need an equation of state $P = P(\rho, T)$. Usually adopt the ideal gas law $P=nkT$ or $P = (\gamma-1)e$ where for monoatomic gas (H), $\gamma=5/3$

Linear Waves

The MHD equations are a set of hyperbolic conservation laws.

An important characteristic of hyperbolic PDEs is they admit wave-like solutions of the form:

$$a = a_0 + \delta a = a_0 + a_1 \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$$

Where a_0 = constant and $a_1/a_0 << 1$. That is, small amplitude perturbations propagate as waves.

To see this (see also Matt's and Muni's notes):

- Substitute this form of solution into MHD equations
- Drop terms nonlinear in δa

Write resulting system of linear equations for perturbations $\mathbf{W} = \delta a_i$ as

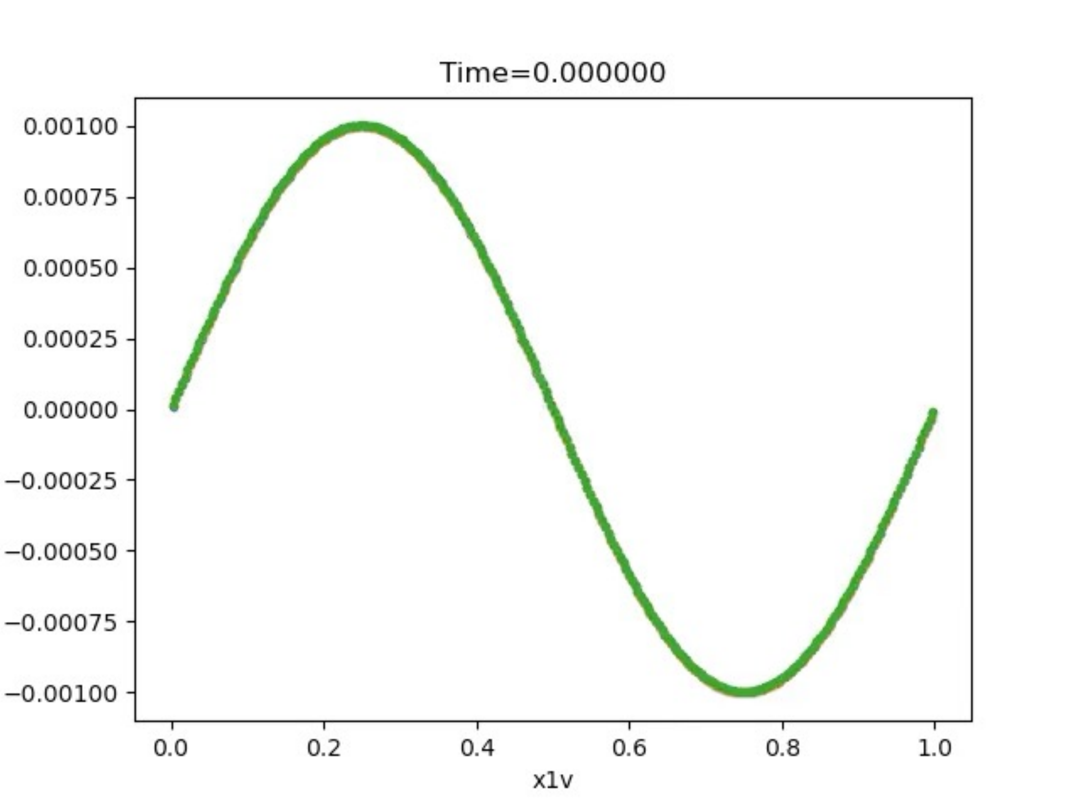
$$\frac{\partial \mathbf{W_1}}{\partial t} + \mathsf{A}(\mathbf{W_0})\frac{\partial \mathbf{W_1}}{\partial x} = 0$$

Each of the eigenvalues and eigenvectors of A correspond to different wave modes in the system, with the eigenvalues given by the characteristic equation (dispersion relation) equal to the phase velocity.

Pure hydrodynamics (no magnetic field) modes are:

- sound waves propagating at (v_0 +/- C_s)
- entropy mode propagating at v_0

ρ, P, v in propagating sound wave



Note: numerical solution computed with grid-based compressible MHD code (AthenaK). Discuss numerical methods next lecture.

Dispersion relation for MHD waves.

Dispersion relation for MHD waves is much more complicated:

$$[\omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2][\omega^4 - \omega^2 k^2 (v_A^2 + C^2) + k^2 C^2 (\mathbf{k}\cdot\mathbf{v}_A)^2] = 0$$

Where $\mathbf{v}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$ is the Alfven speed

$C^2 = \gamma P_0/\rho_0$ is the sound speed

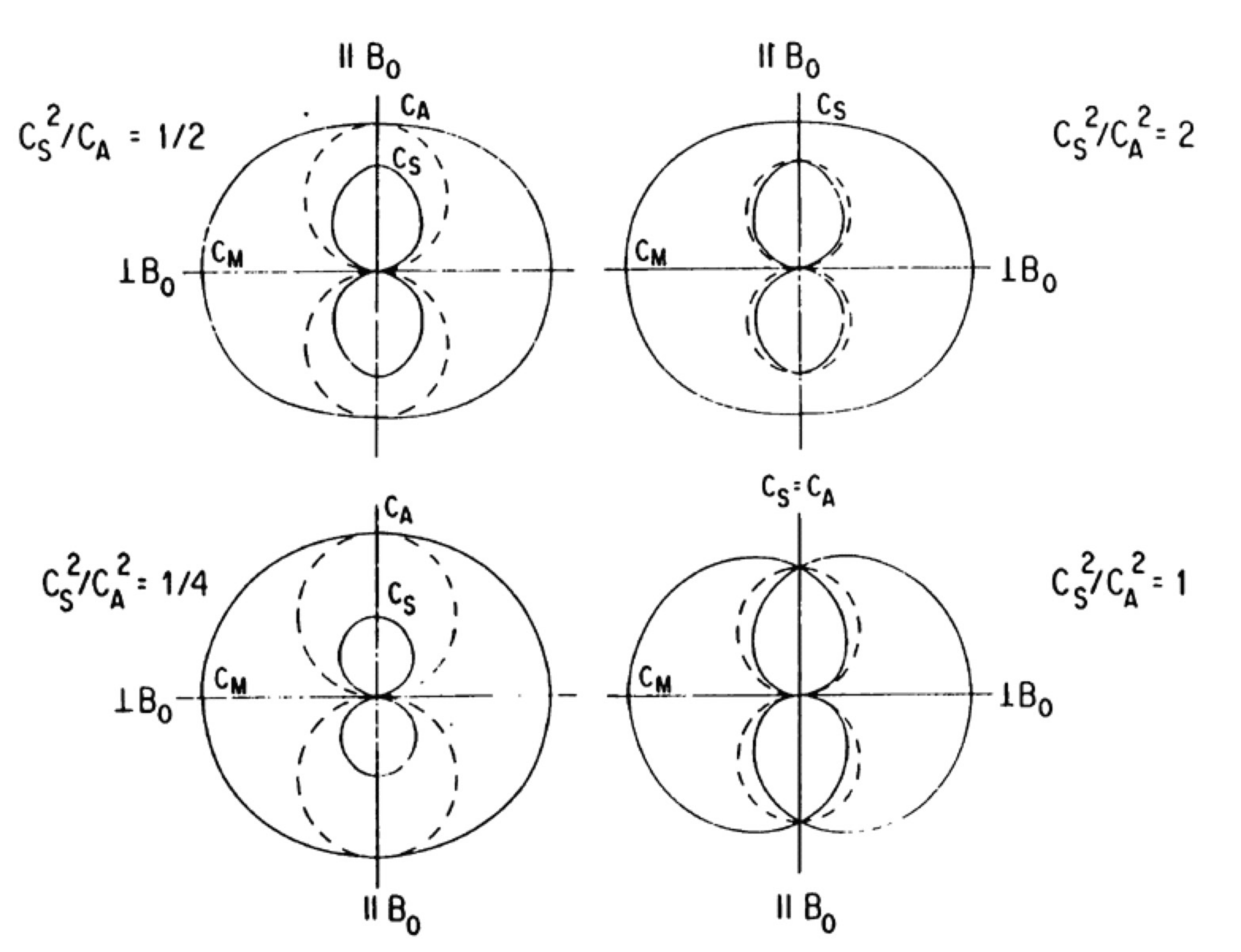
In MHD there are three wave families (in addition to the entropy mode). Note there is only one in hydrodynamics!:

1) Alfven wave propagates at V_A

2) and 3) Fast and slow magnetosonic waves propagating at $C_{f,s}$

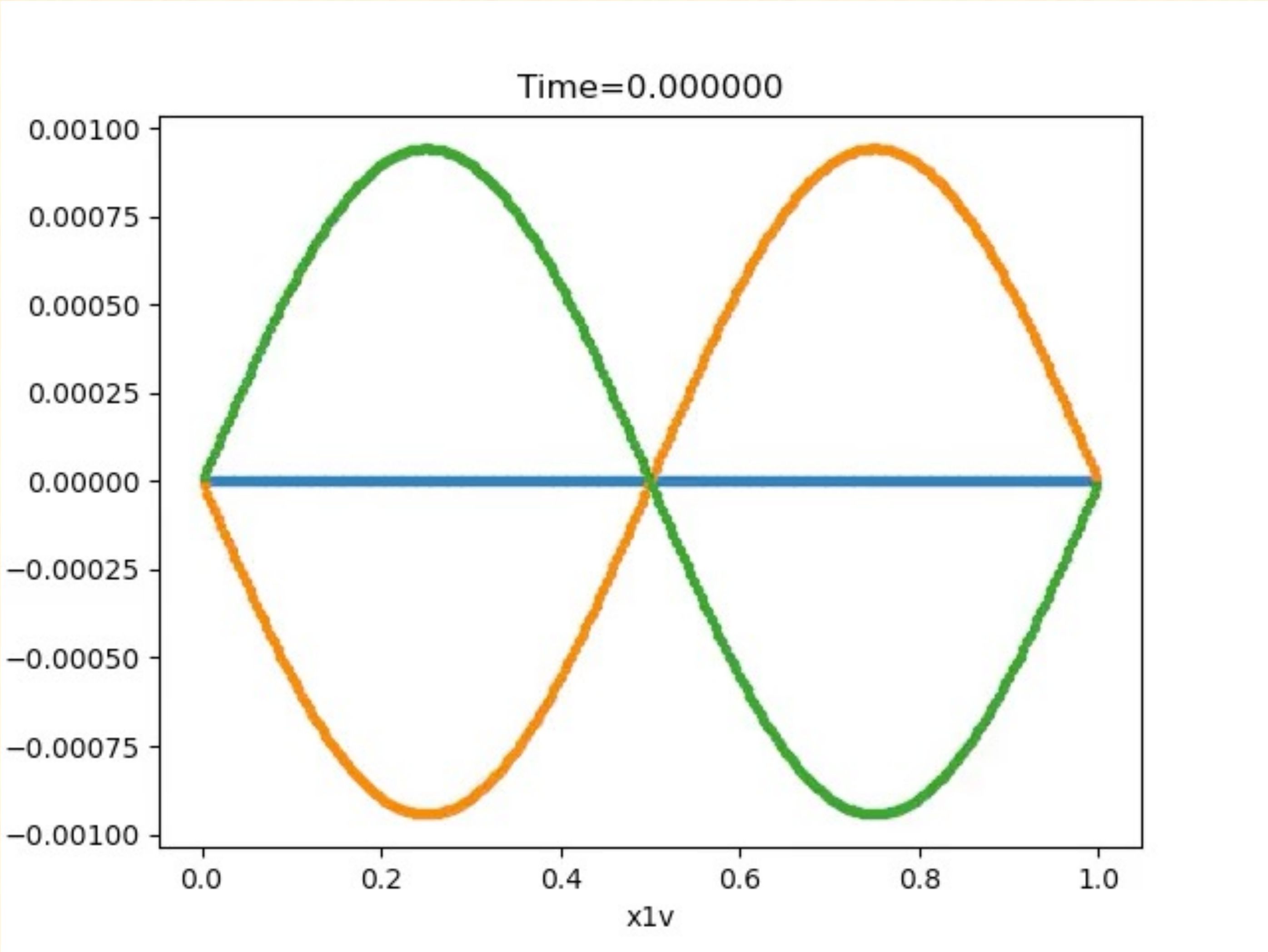
$$C_{f,s}^2 = \frac{1}{2}\left([C^2 + v_A^2] \pm \sqrt{[C^2 + v_A^2]^2 - 4C^2 v_{Ax}^2}\right)$$

Phase velocities of MHD waves: Friedrichs diagrams.

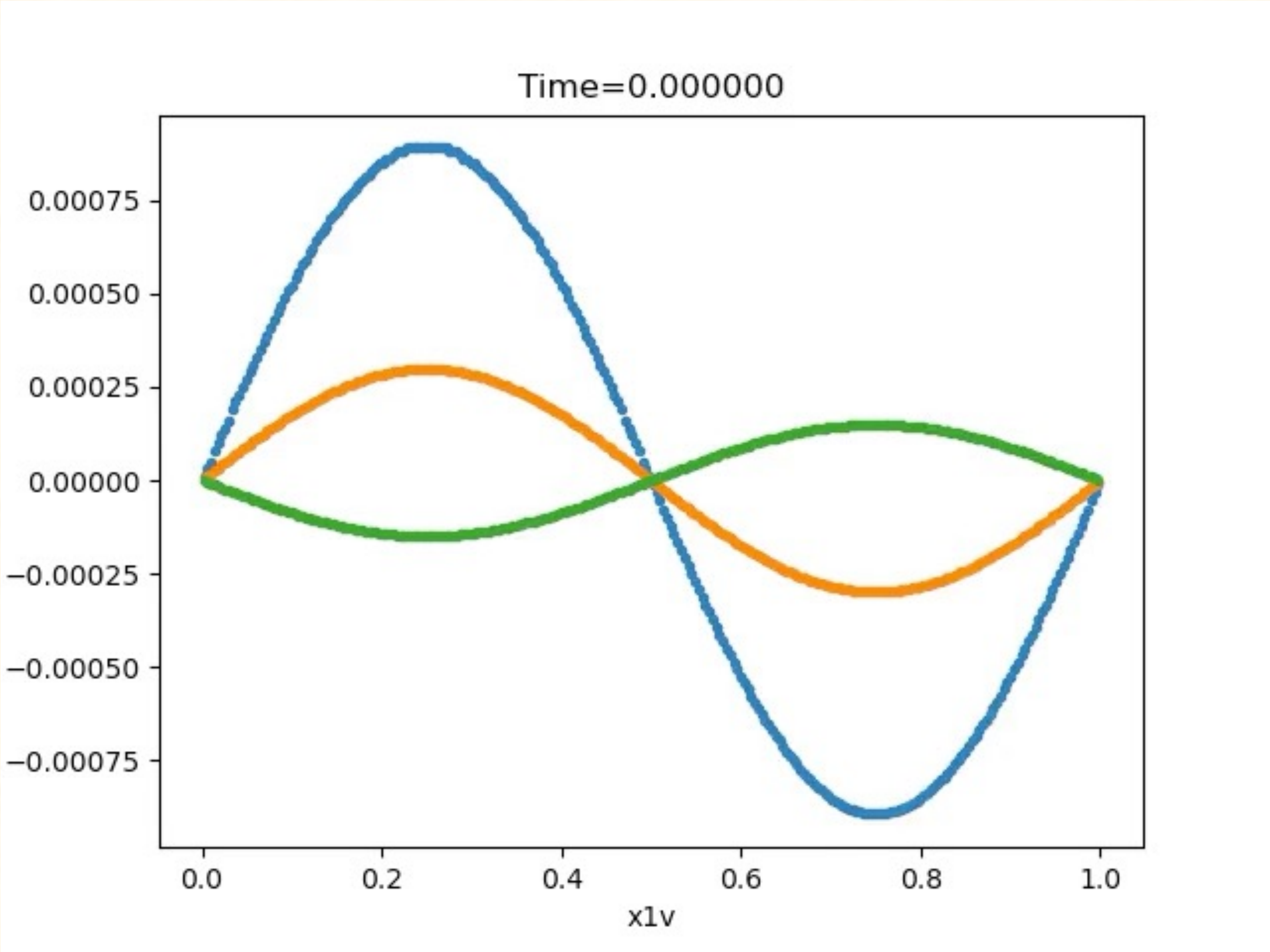


Note for in some cases, modes are degenerate. Eigenvalues of linearized MHD equations are not always linearly independent. MHD equations are not *strictly hyperbolic*.

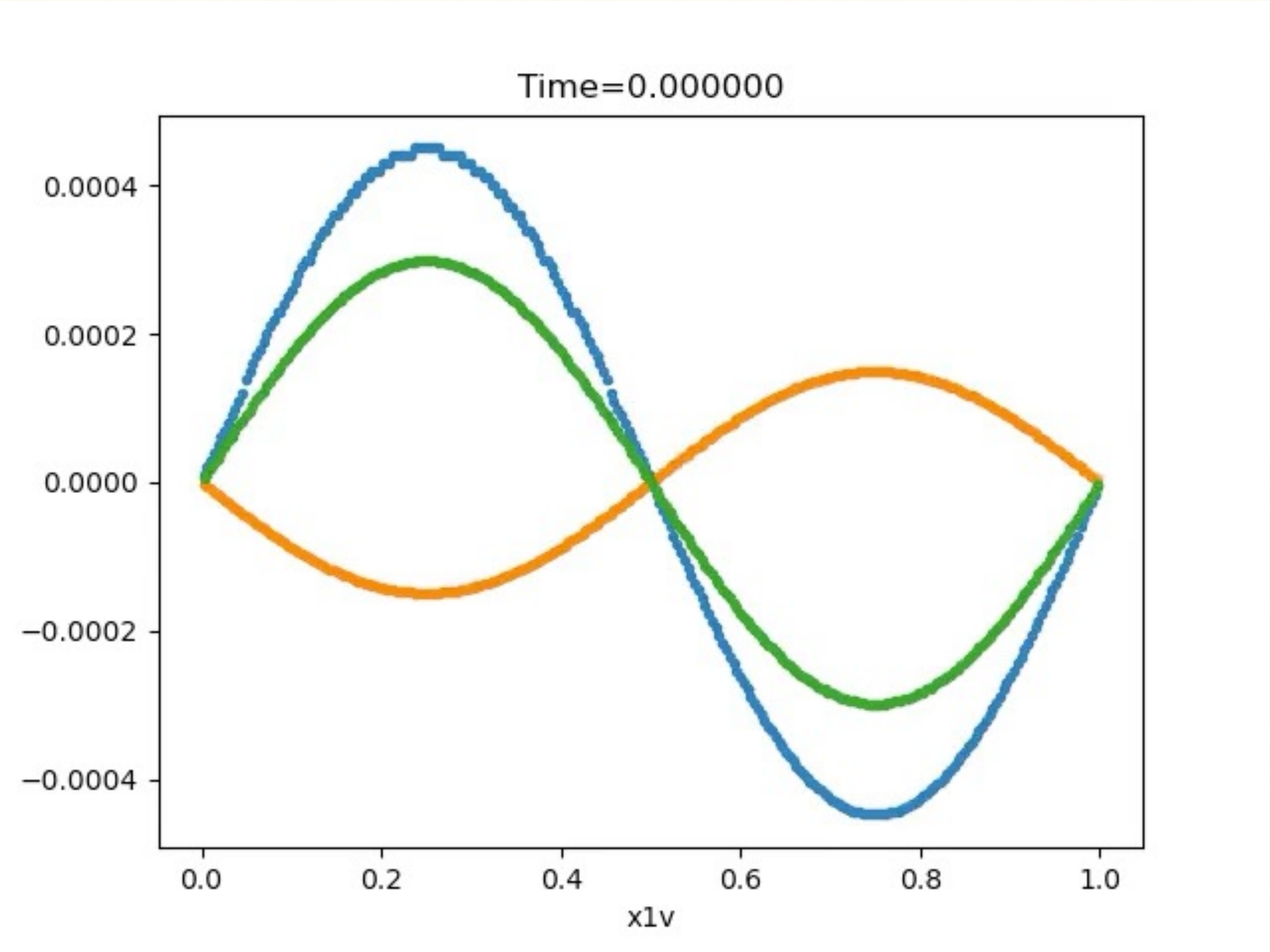
ρ (blue), $B_\perp$ (green), $v_\perp$ (orange) in propagating Alfven wave



ρ (blue), $B_\perp$(green), $v_\perp$(orange) in propagating slow magnetosonic wave



ρ (blue), $B_\perp$ (green), $v_\perp$ (orange) in propagating fast magnetosonic wave



Linear Instabilities

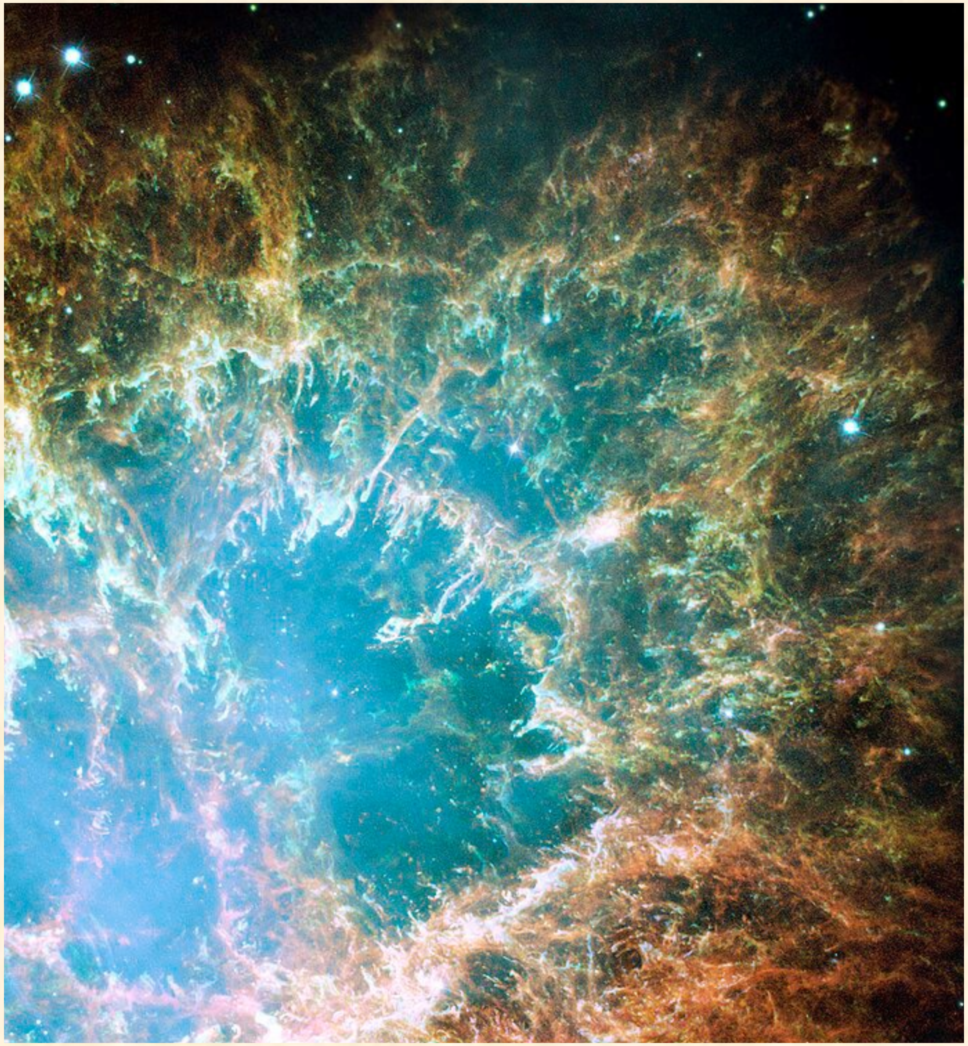
Much of the complexity in astrophysical fluids is a result of the nonlinear saturation of MHD instabilities.

We'll talk about three fundamental fluid instabilities here (there are many more):

1. Rayleigh-Taylor (RT) instability.
2. Kelvin-Helmholtz (KH) instability.
3. Magneto-rotational instability (MRI)

See monographs by Chandrasekhar 1965
Drazin & Reid 1981

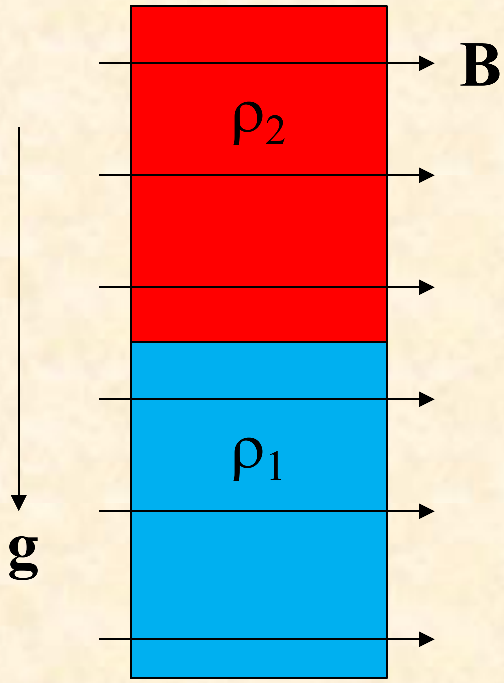
Rayleigh-Taylor (RT) Instability



HST image of Crab nebula

Dispersion relation

Consider a heavy fluid (ρ_2) overtop of a light fluid (ρ_1) in a vertical gravitational field **g** = $-g\hat{z}$ and horizontal magnetic field **B**.



In this system, vertical displacements oscillate with frequency:

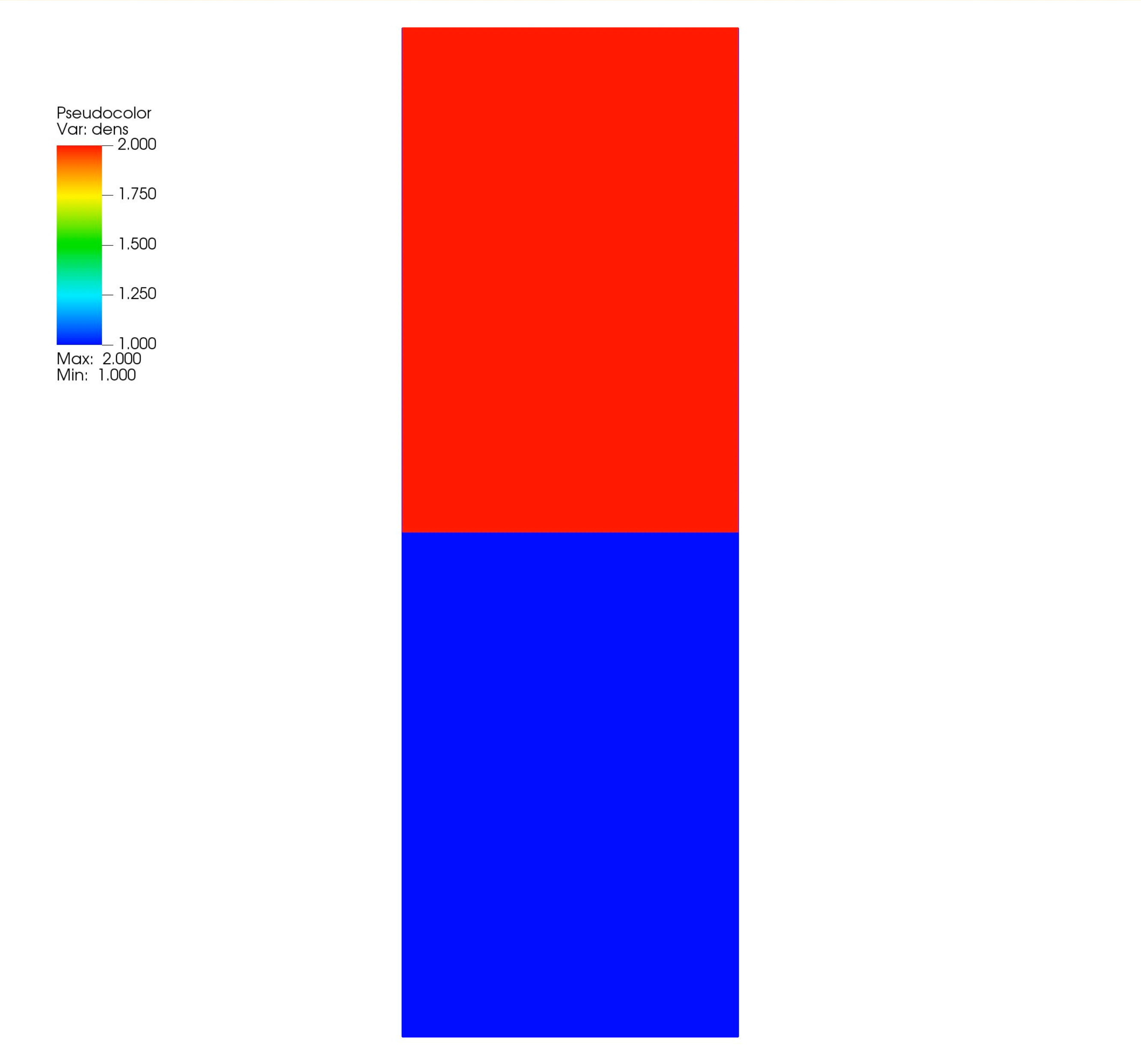
$$\omega = \pm i \left[|k| g \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} + \frac{(\mathbf{k} \cdot \mathbf{B})^2}{2\pi(\rho_1 + \rho_2)} \right]^{1/2}$$

When $\rho_2 > \rho_1$ then $\omega^2 < 0$, $\exp(-i\omega t)$ grows with time: **instability**

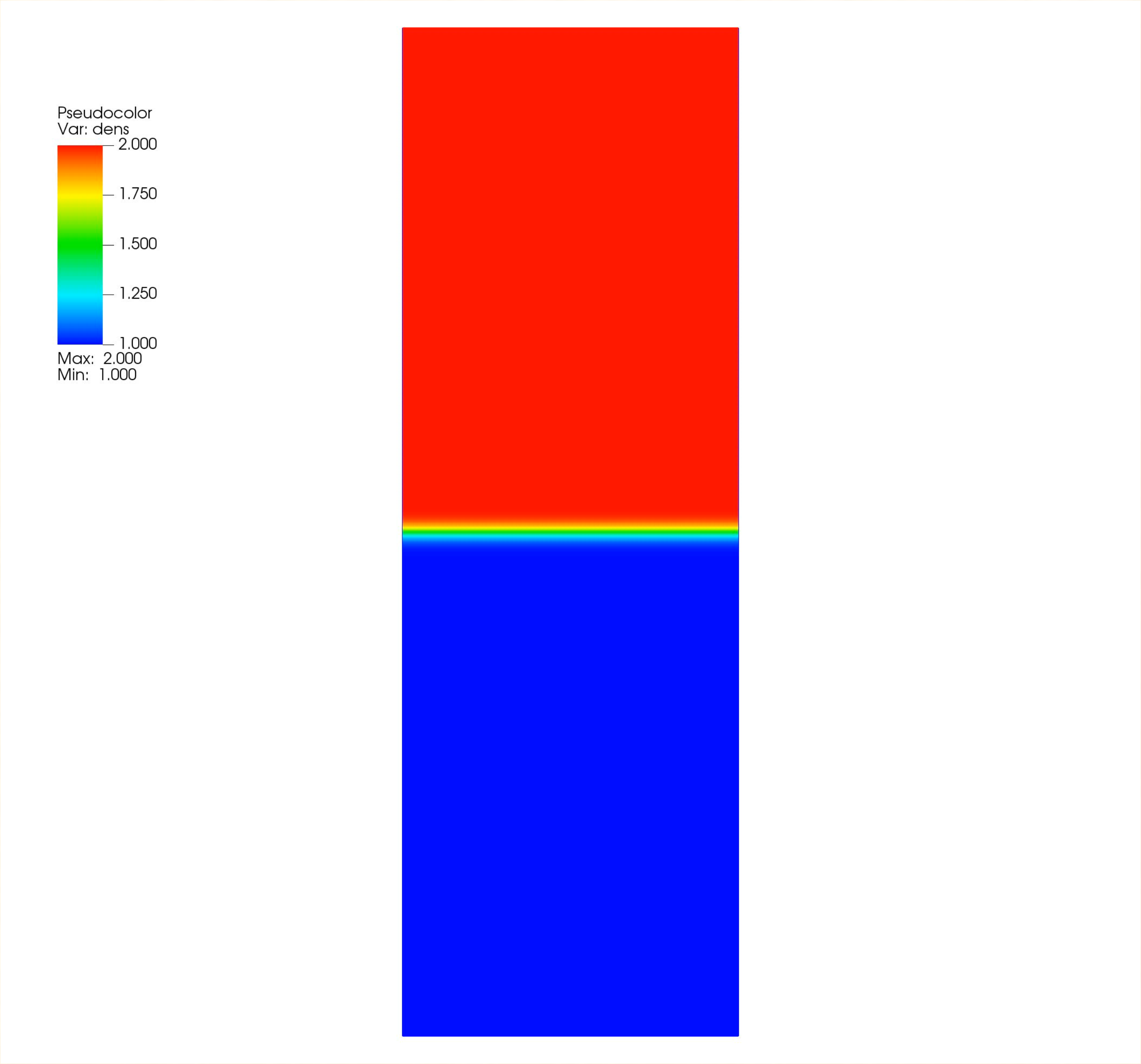
Growth rate depends on Atwood number: $$A = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}$$

Magnetic fields stabilize short wavelength modes.

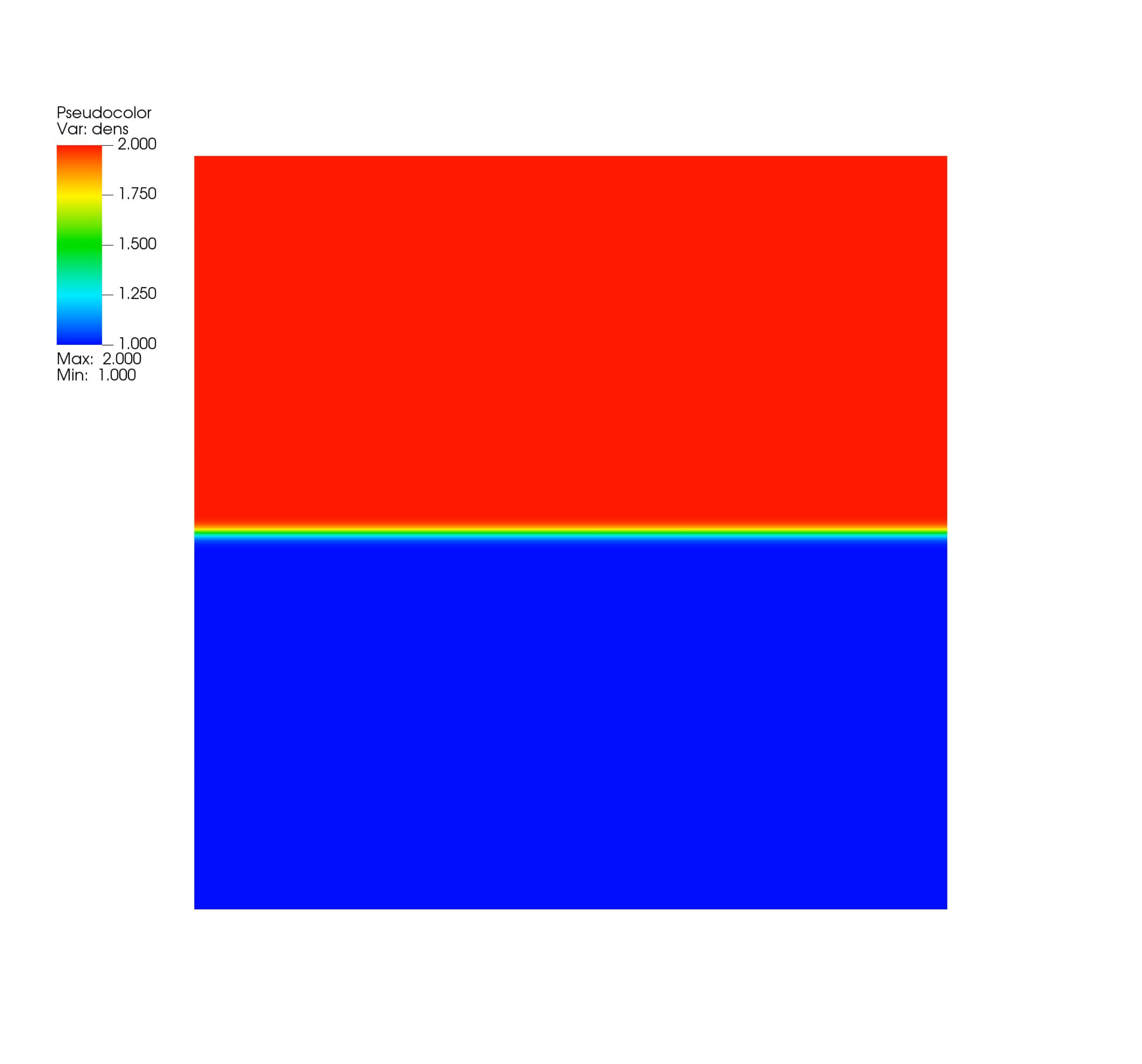
Hydrodynamic evolution of single mode with a contact discontinuity between fluids



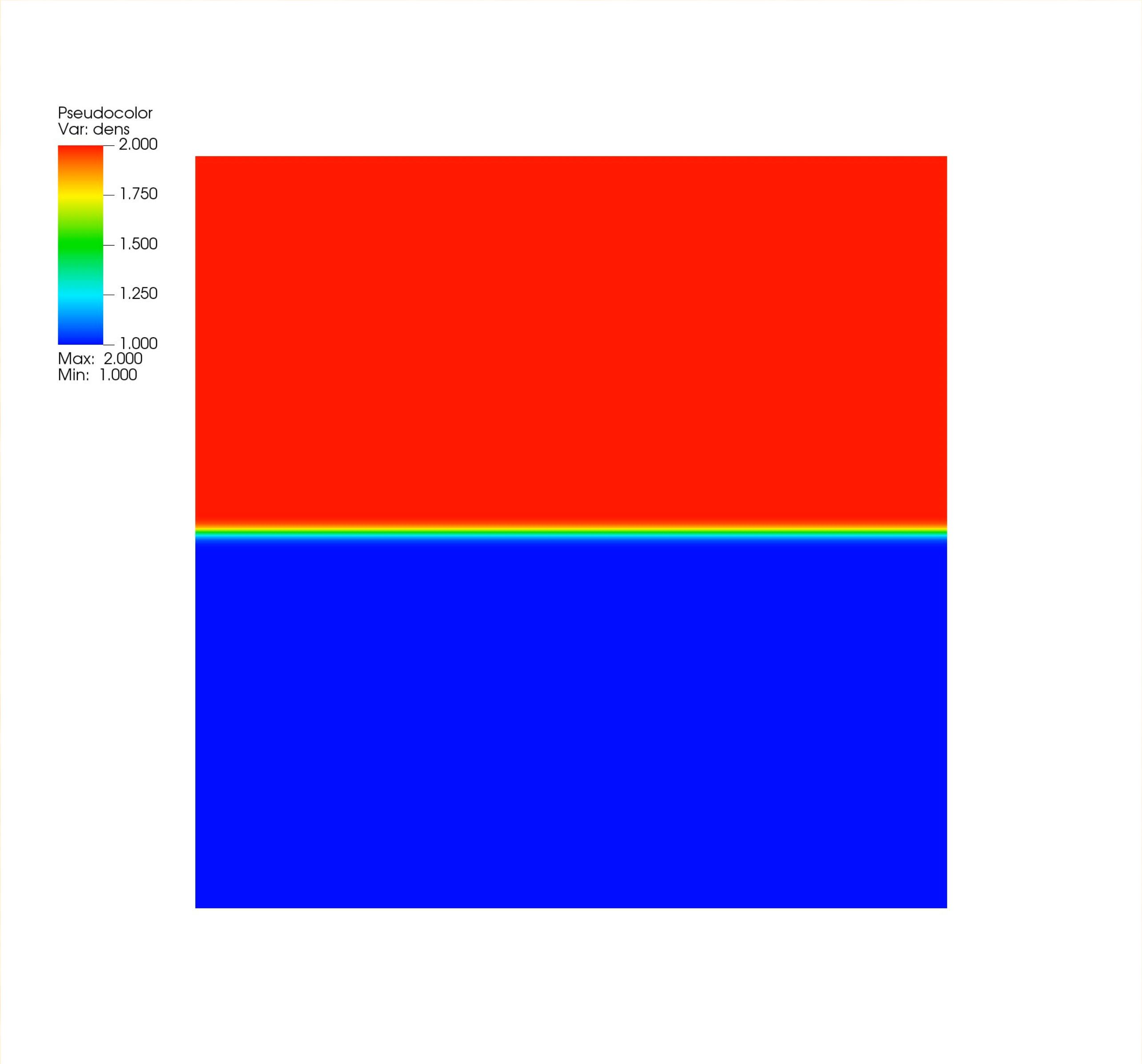
Hydrodynamic evolution of a single mode with a smooth interface (tanh) between fluids



Hydrodynamic evolution with three wavelengths across domain, smooth interface



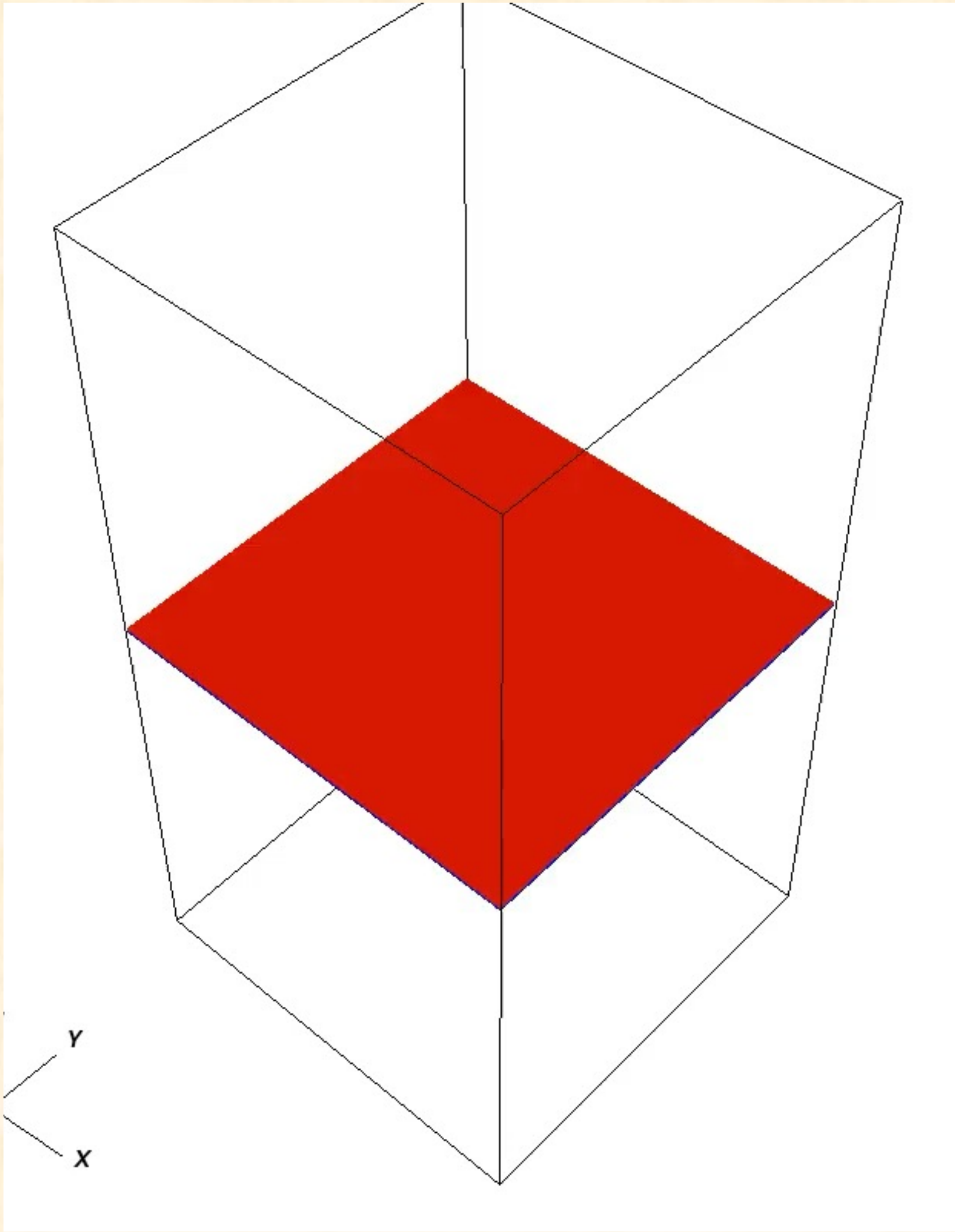
Hydrodynamic evolution with random perturbations and smooth interface



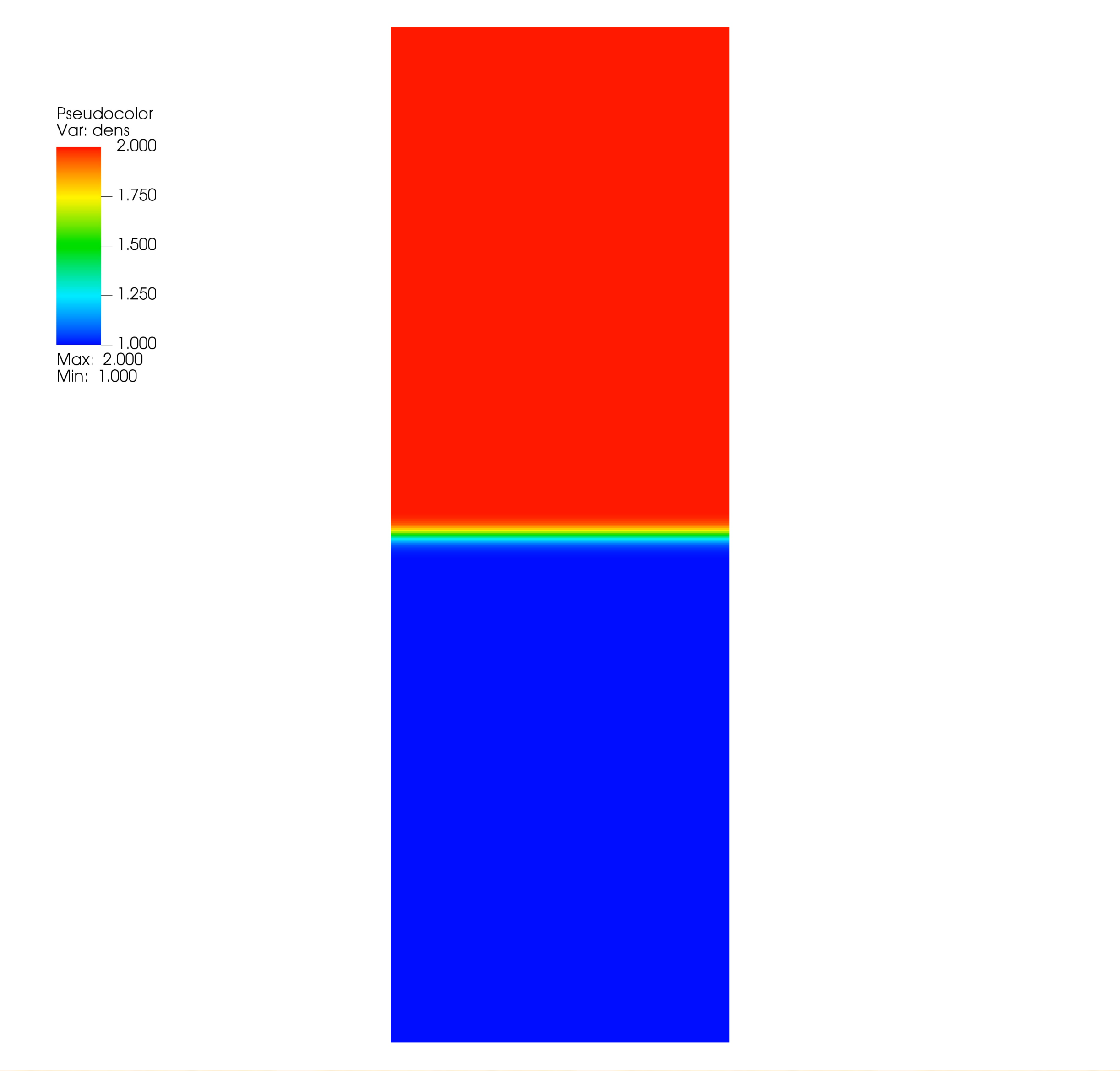
3D Hydro RTI

256^2 x 512 grid
Random perturbations

Isosurface and slices of density



MHD evolution of a single mode with a smooth interface.



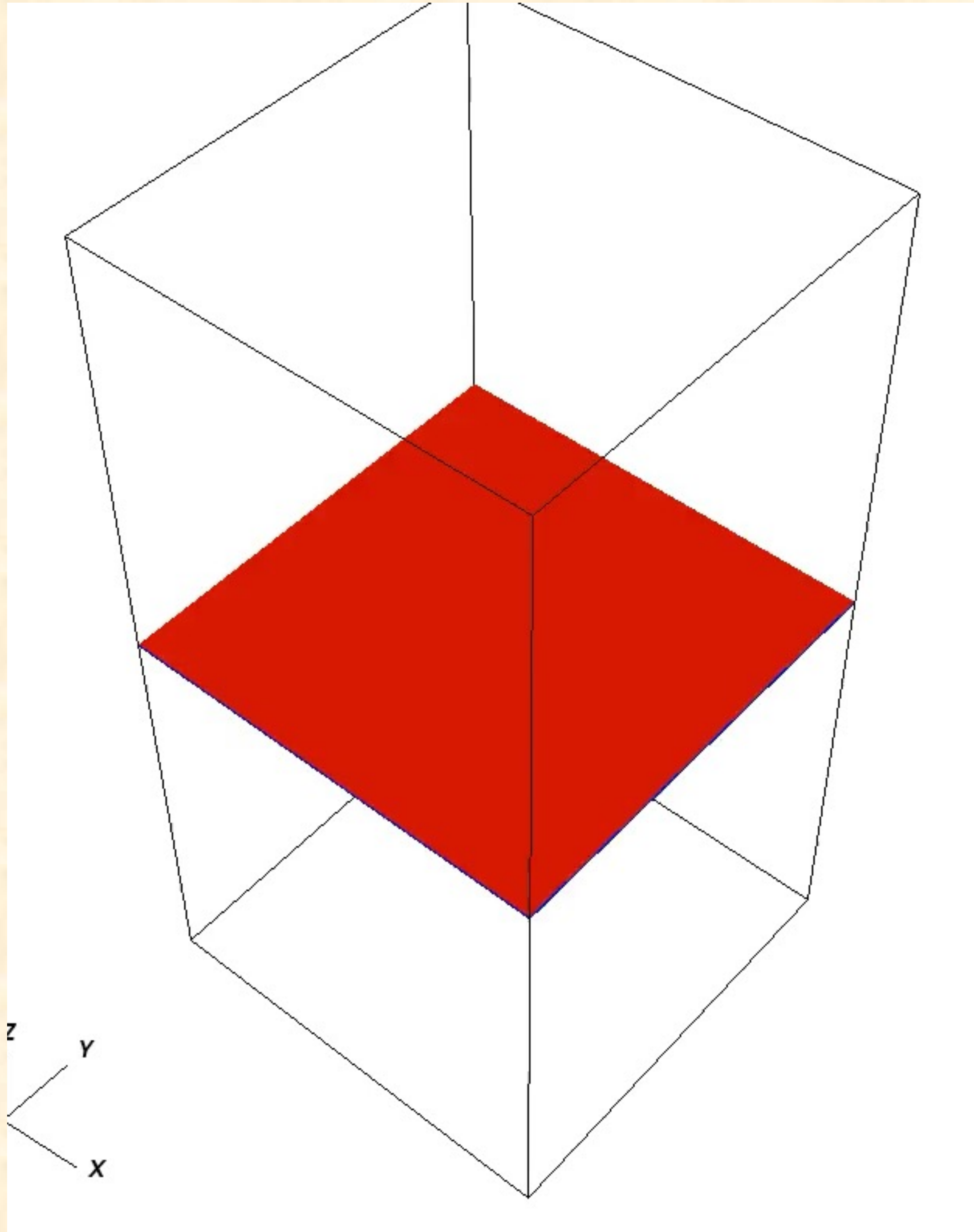
3D MHD RTI

256^2 x 512 grid

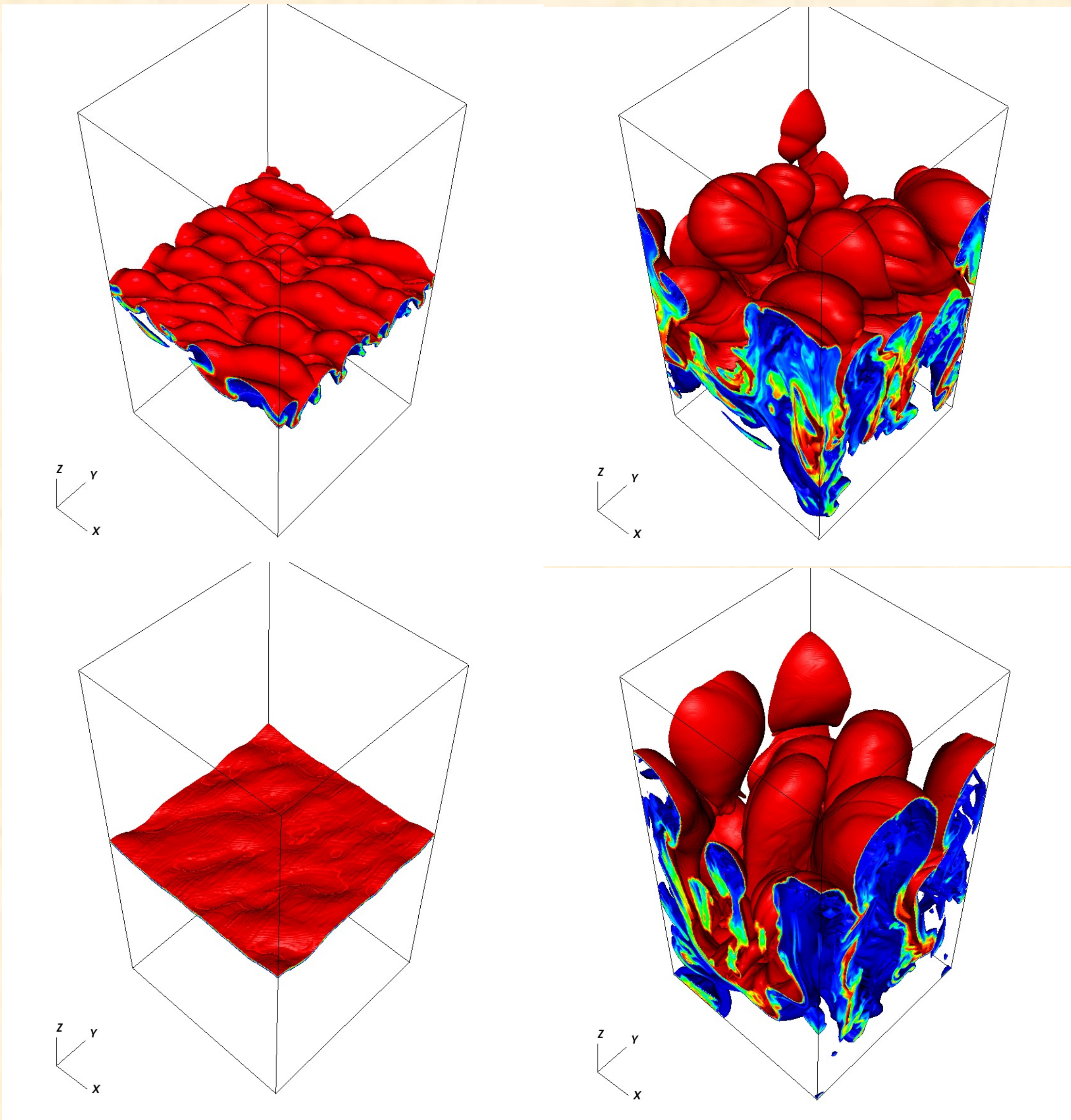
Random perturbations

Isosurface and slices of density

$\mathbf{B} = (B_x, 0, 0)$
$\lambda_{crit} = L_x/3$



Magnetic fields cannot suppress all modes of RTI



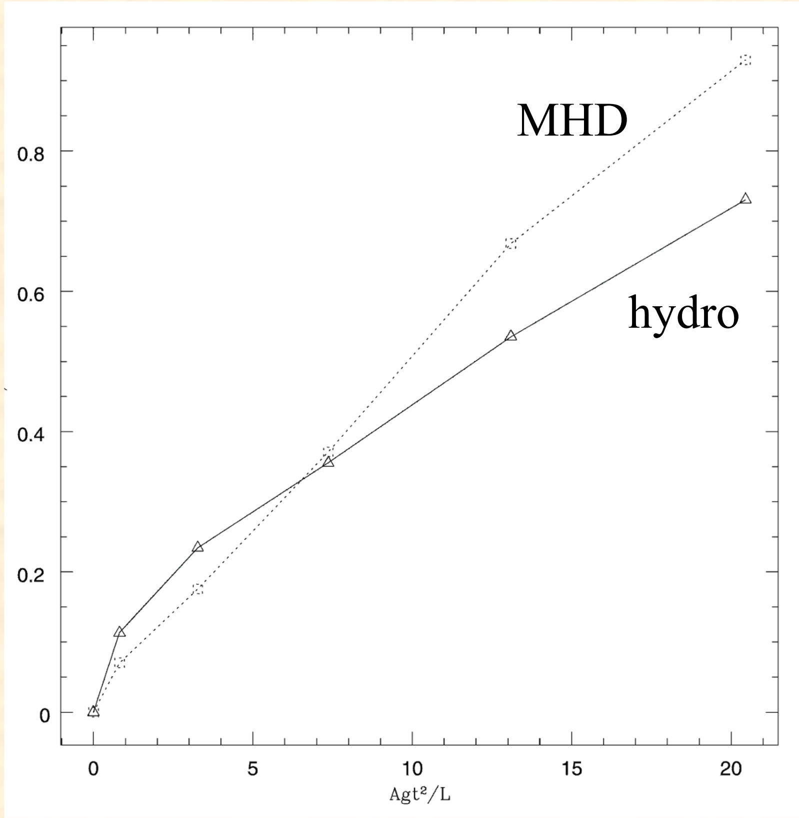
Field rotated by 45^o

Field rotated by 90^o

$t = 20t_s$ $t = 40t_s$

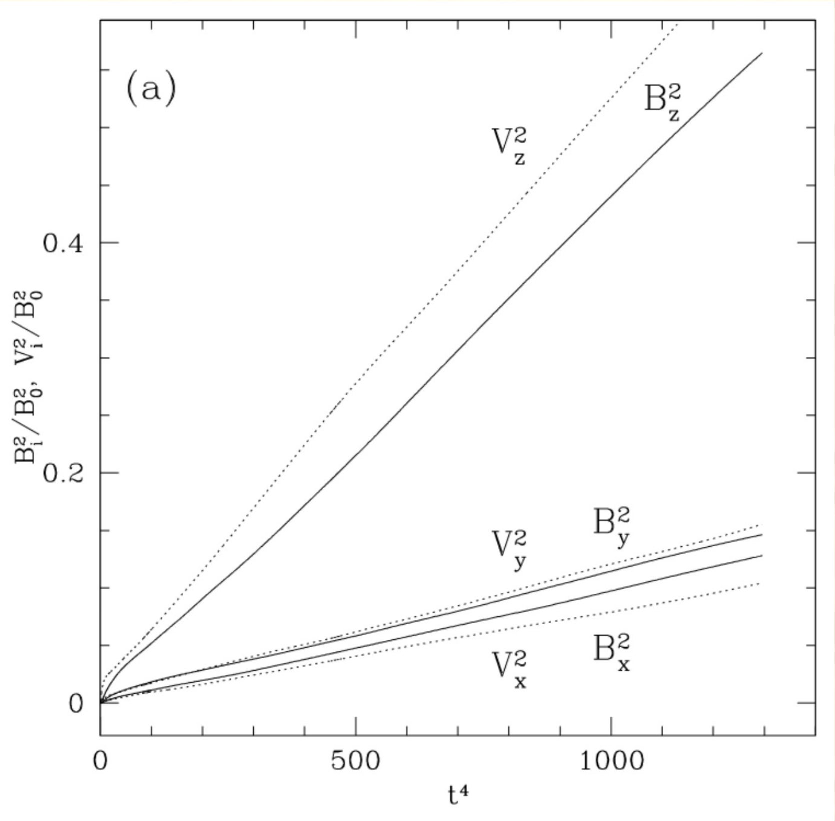
Simple scaling arguments can be applied to growth of bubbles in nonlinear regime

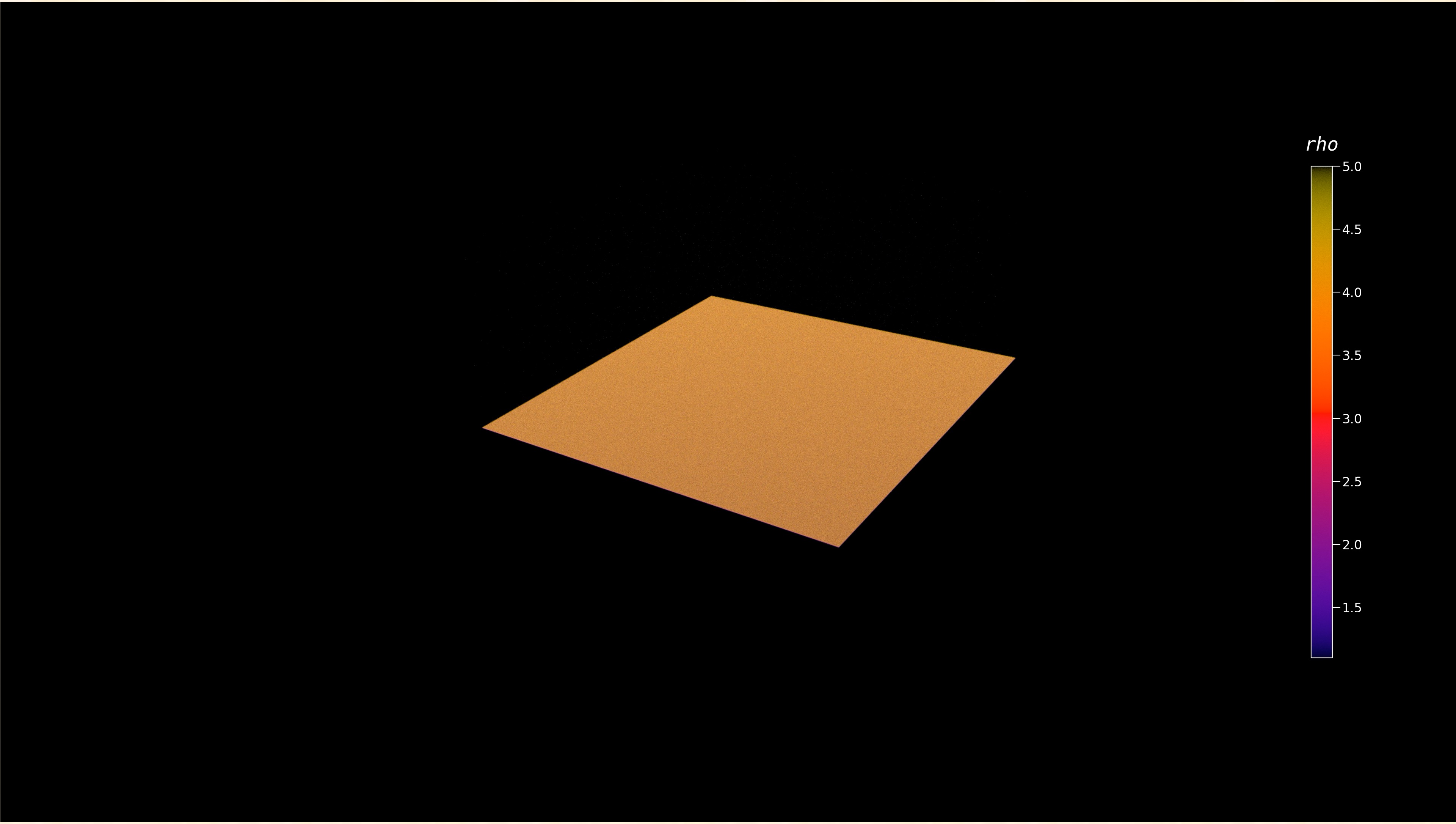
Expect height of the bubbles: $h \sim gt^2$



Bubbles rise faster in MHD!

Expect energy released $\sim h^2 \sim t^4$





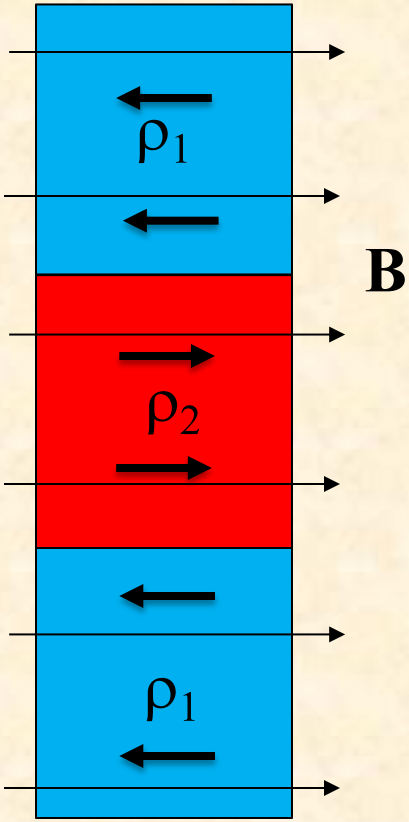
Largest MHD simulation to date, 5120^3 grid, 150M cpu hours, 400TB of data (courtesy Chuanfei Dong, BU).
Study turbulence and dynamo in RTI.

Kelvin-Helmholtz (KH) Instability

Jupiter's
Great Red
Spot

Dispersion relation

Consider two fluids (ρ_2 and ρ_1) in relative (shear) motion, with a horizontal magnetic field **B**.



In this system, vertical displacements oscillate with frequency:

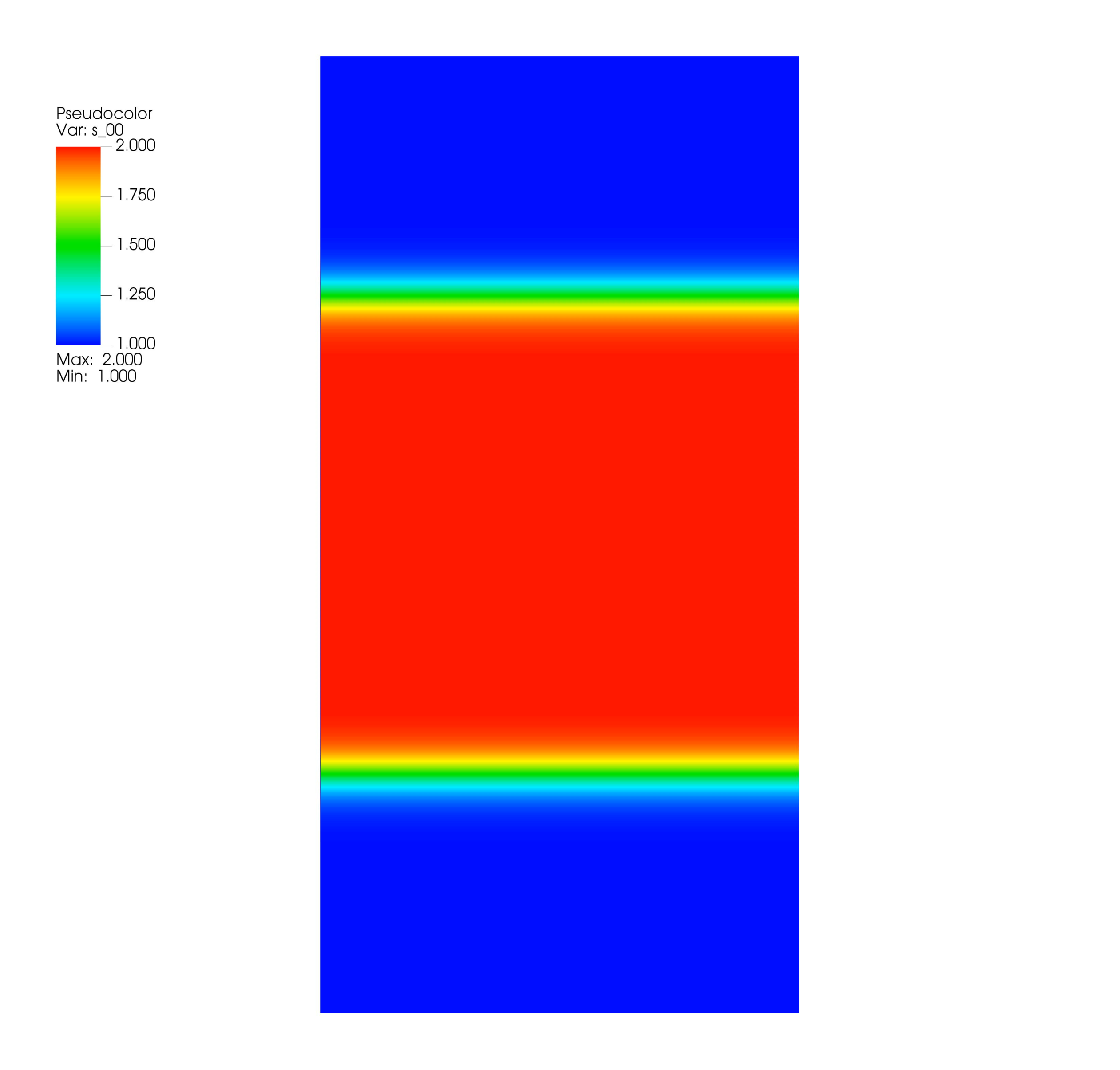
$$\omega = \frac{k_x U}{2}\frac{\bar{\rho}}{\rho_1}\left[1 \pm i\sqrt{\frac{\rho_1}{\rho_2}\left\{1 - \frac{(\mathbf{k}\cdot\mathbf{B})^2}{\pi\rho k_x^2 U^2}\right\}}\right]$$

$$\bar{\rho} = 2\rho_1\rho_2/(\rho_1+\rho_2)$$

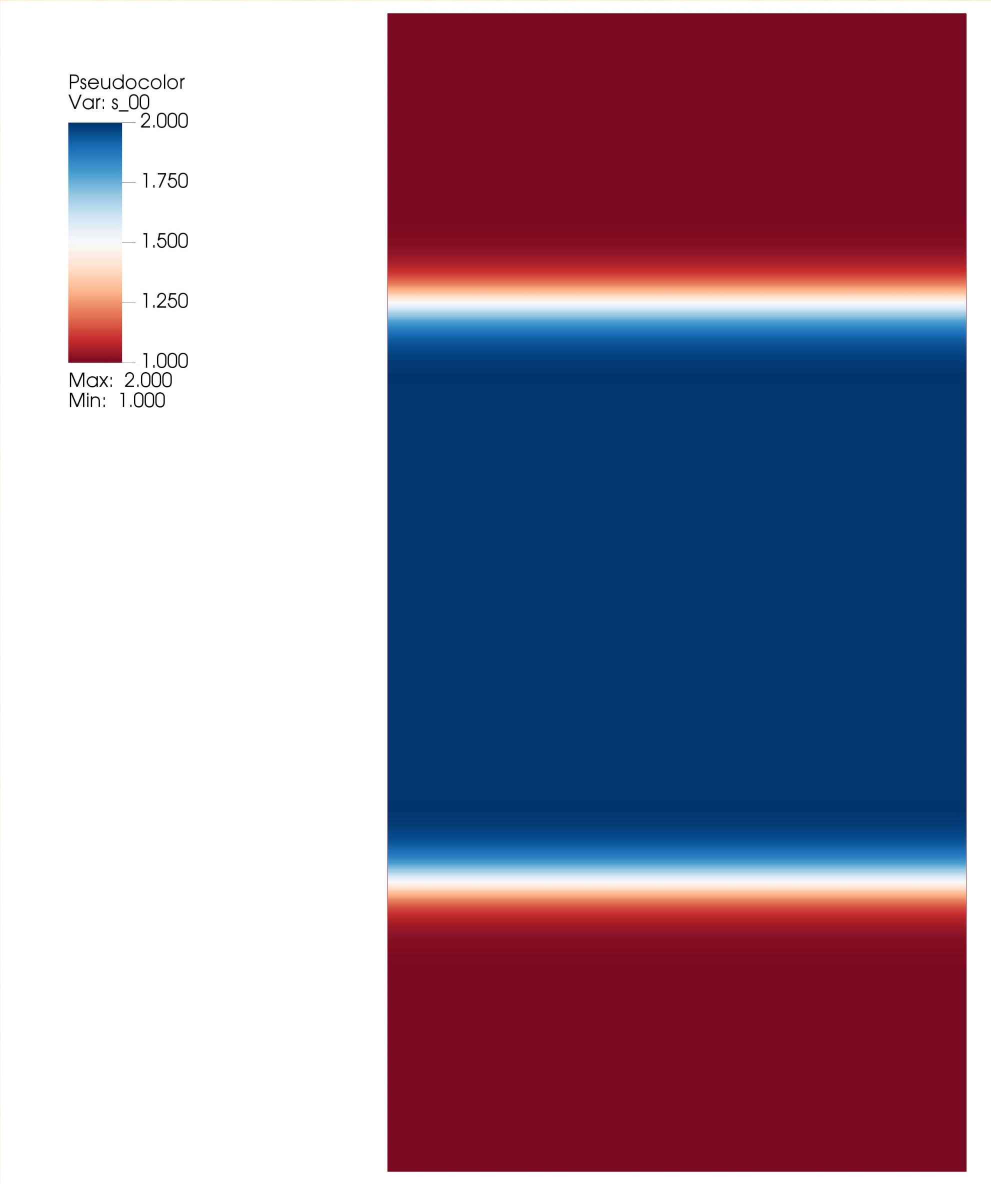
When $U > 0$ then ω is complex, imaginary part gives growth with time: **instability**

Magnetic fields stabilize short wavelength modes.

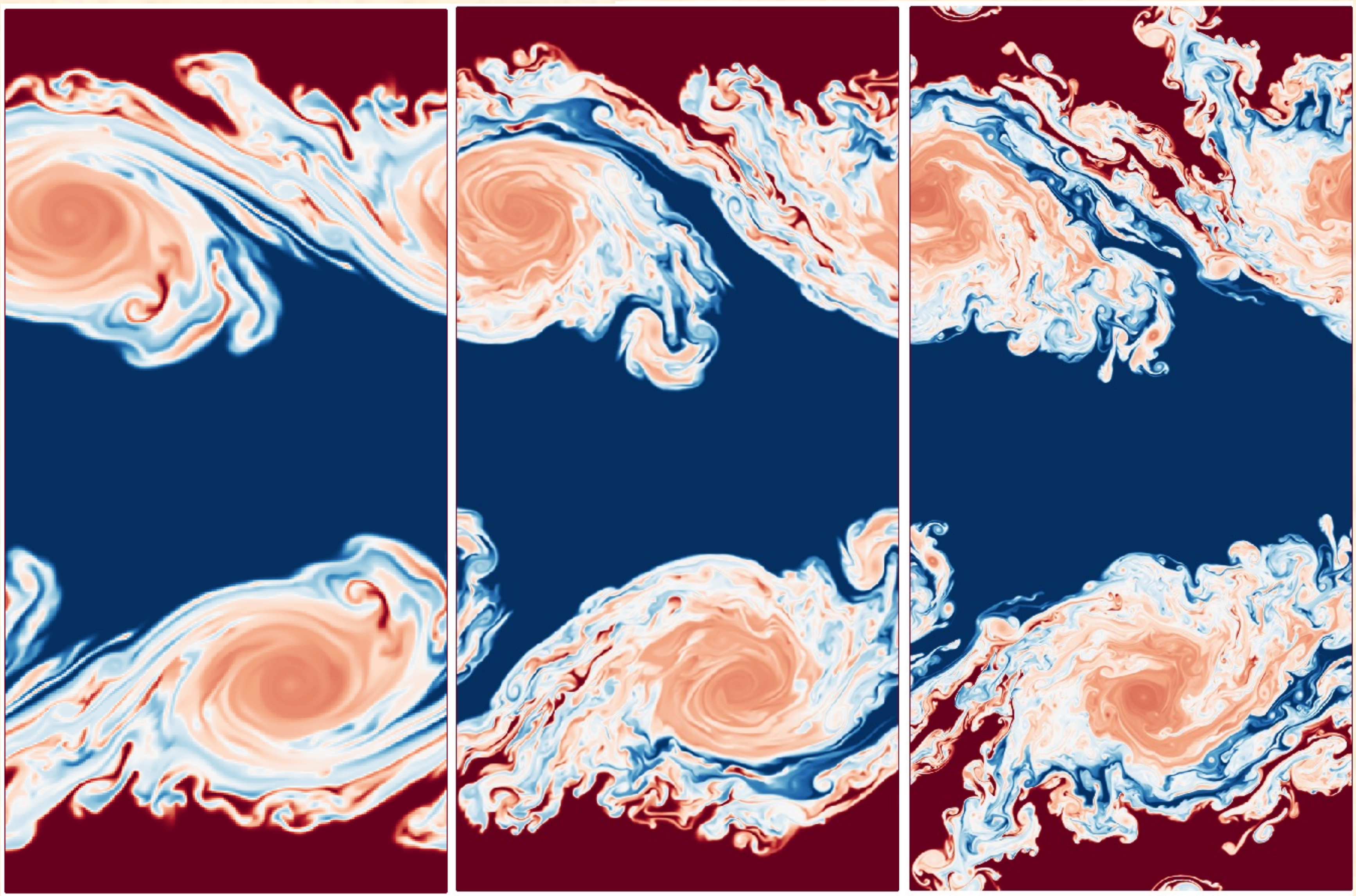
Hydrodynamic evolution of a passive scalar in two fluids with the same density



Hydrodynamic evolution of a passive scalar in two fluids with different densities

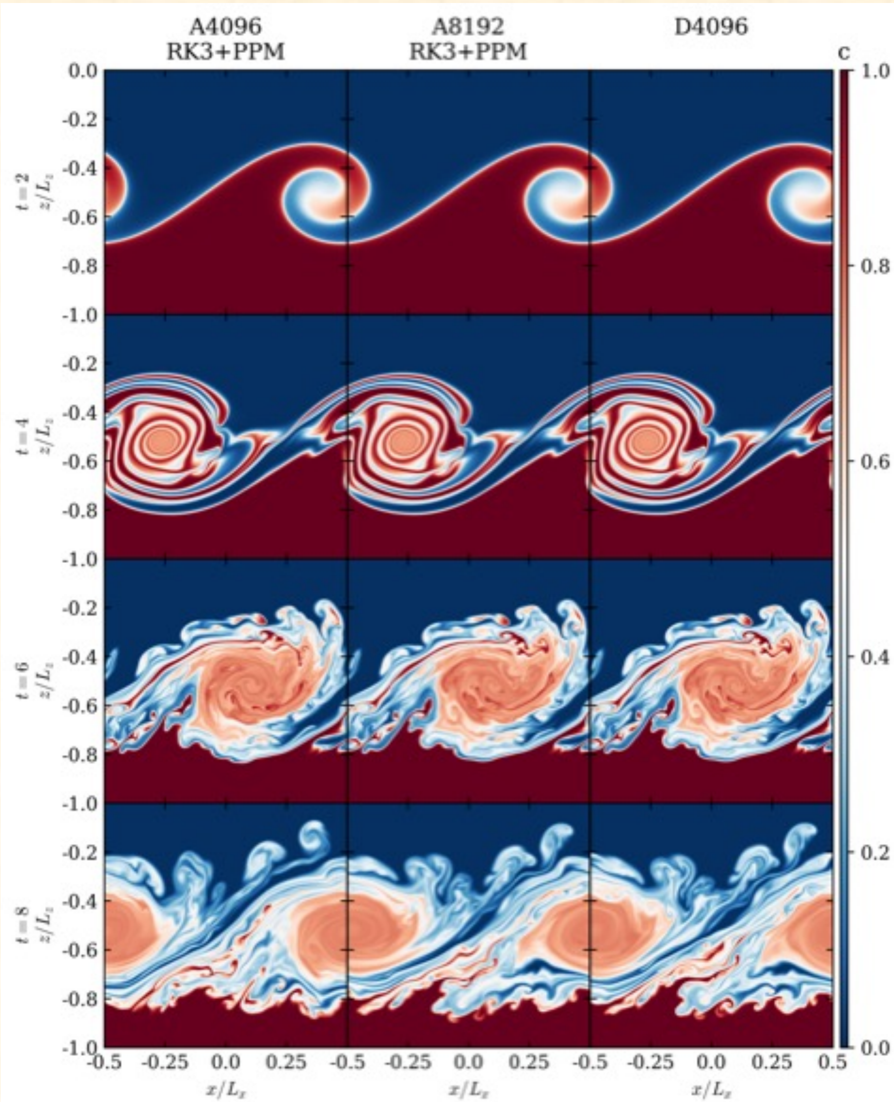


With no viscosity, ever smaller structures with ever higher resolution.
No convergence.



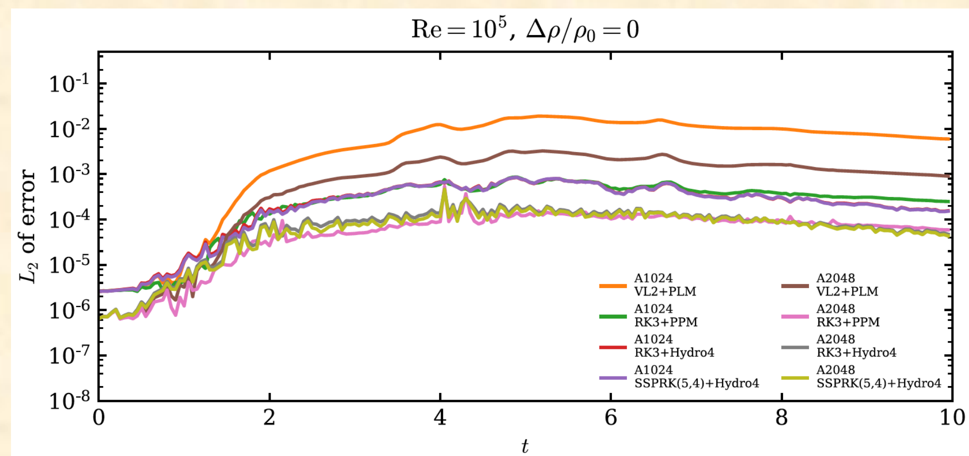
128x256 256x512 512x1024

Convergence requires explicit dissipation



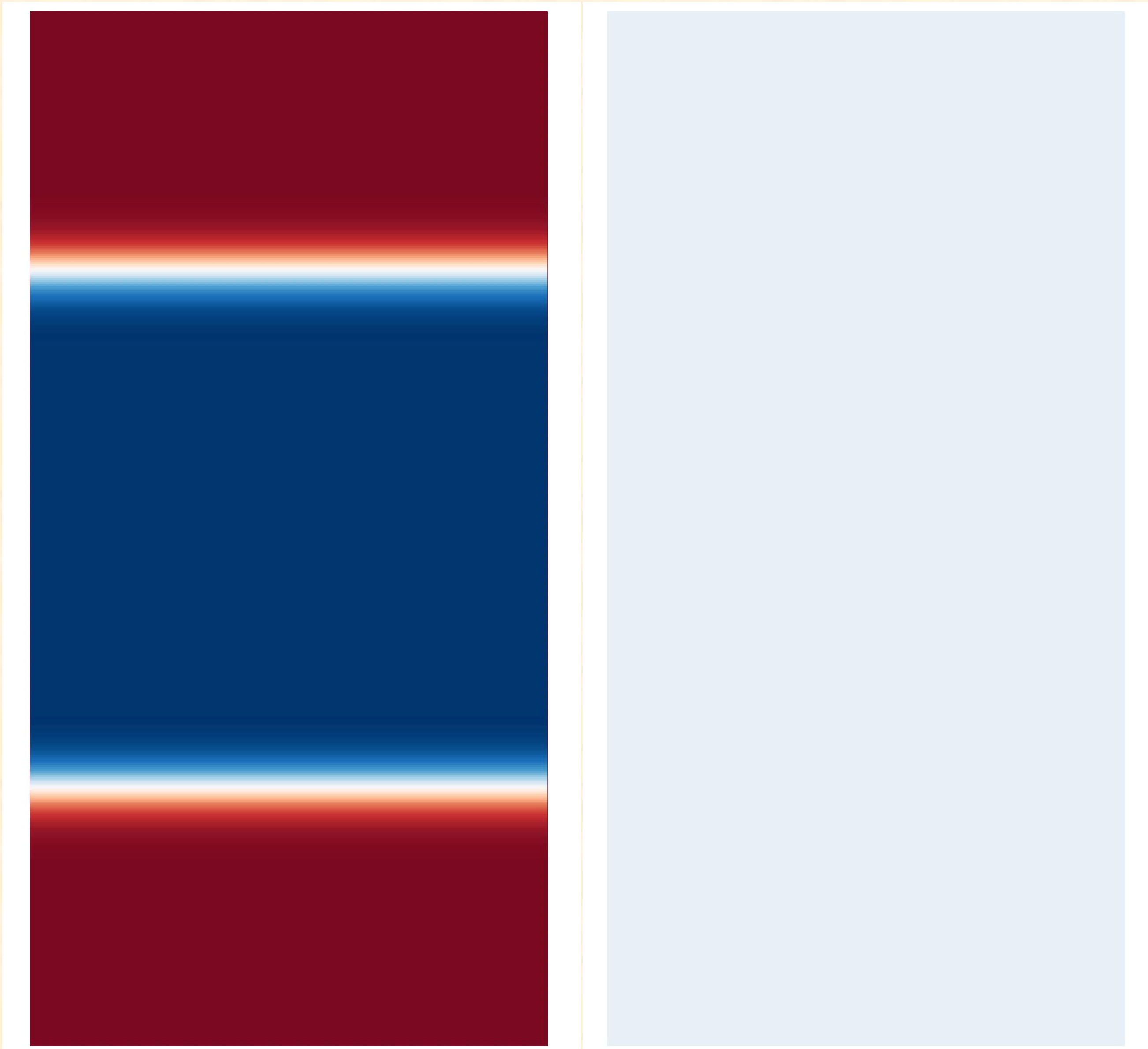
Add explicit viscosity and heat conduction. Then all codes converge to same reference solution.

Athena++ reproduces spectral code results at 2x resolution and 1/2 the cost.



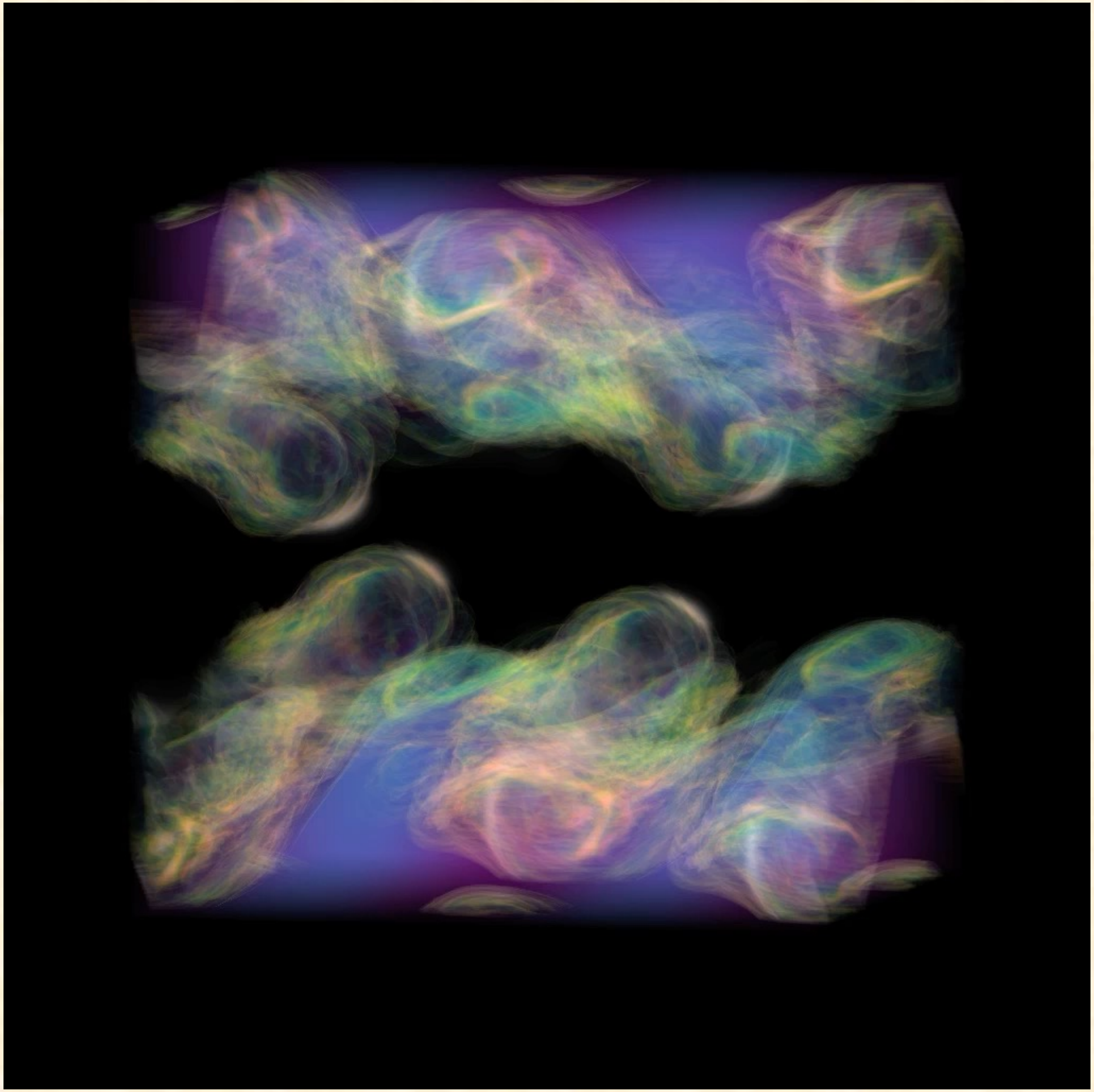
Lecoanet+ 2016

Nonlinear evolution of KHI in 2D MHD

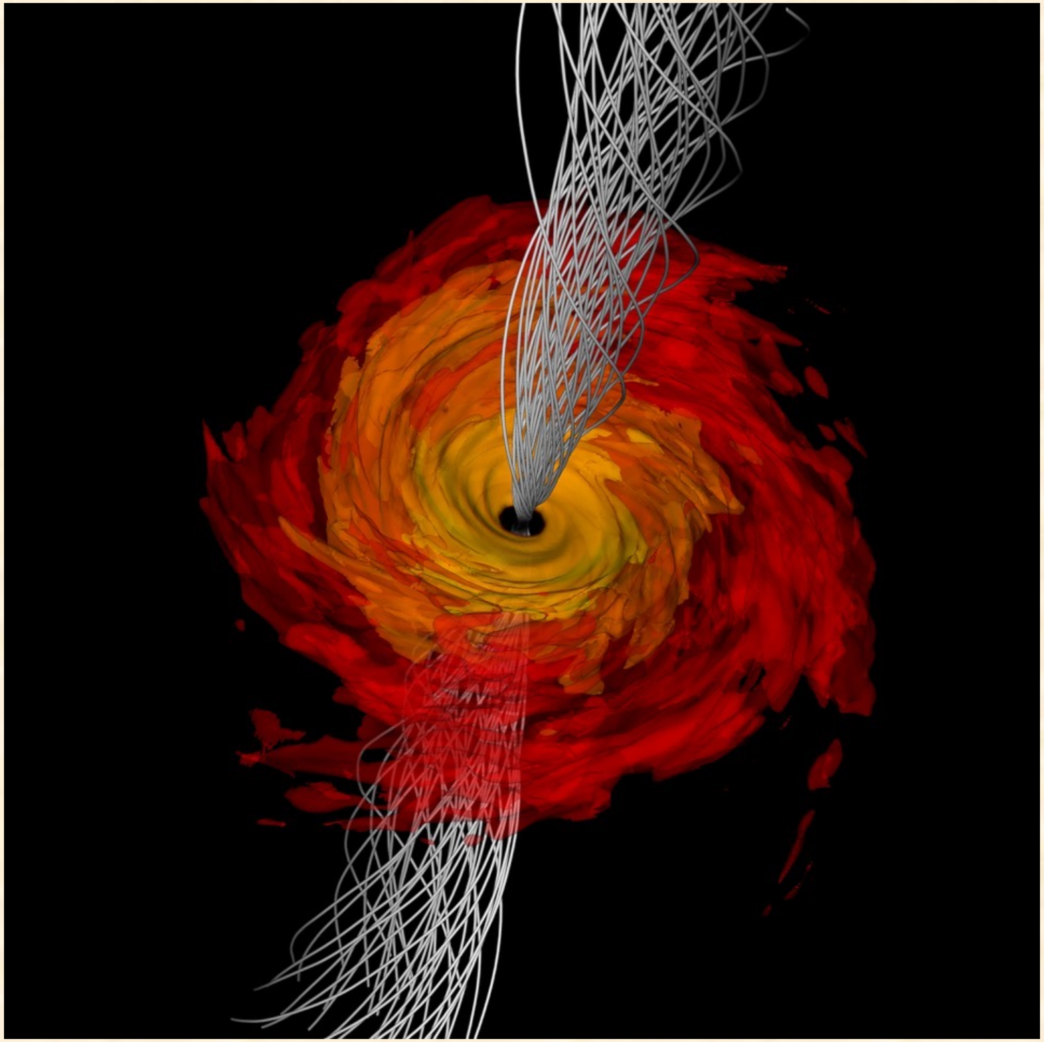


Passive scalar B_x

Structure of KH rolls in 3D MHD



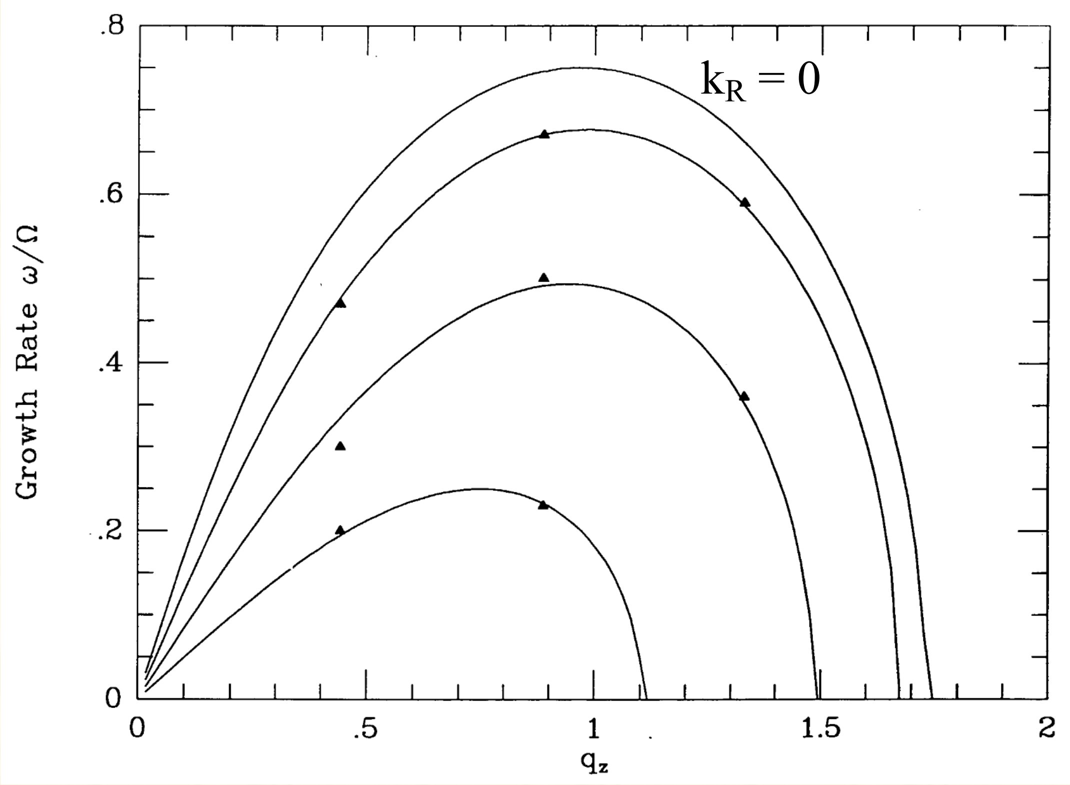
Magneto-rotational Instability (MRI)



Important for *angular momentum transport* in accretion disks.

Dispersion relation

Different lines are different k_R

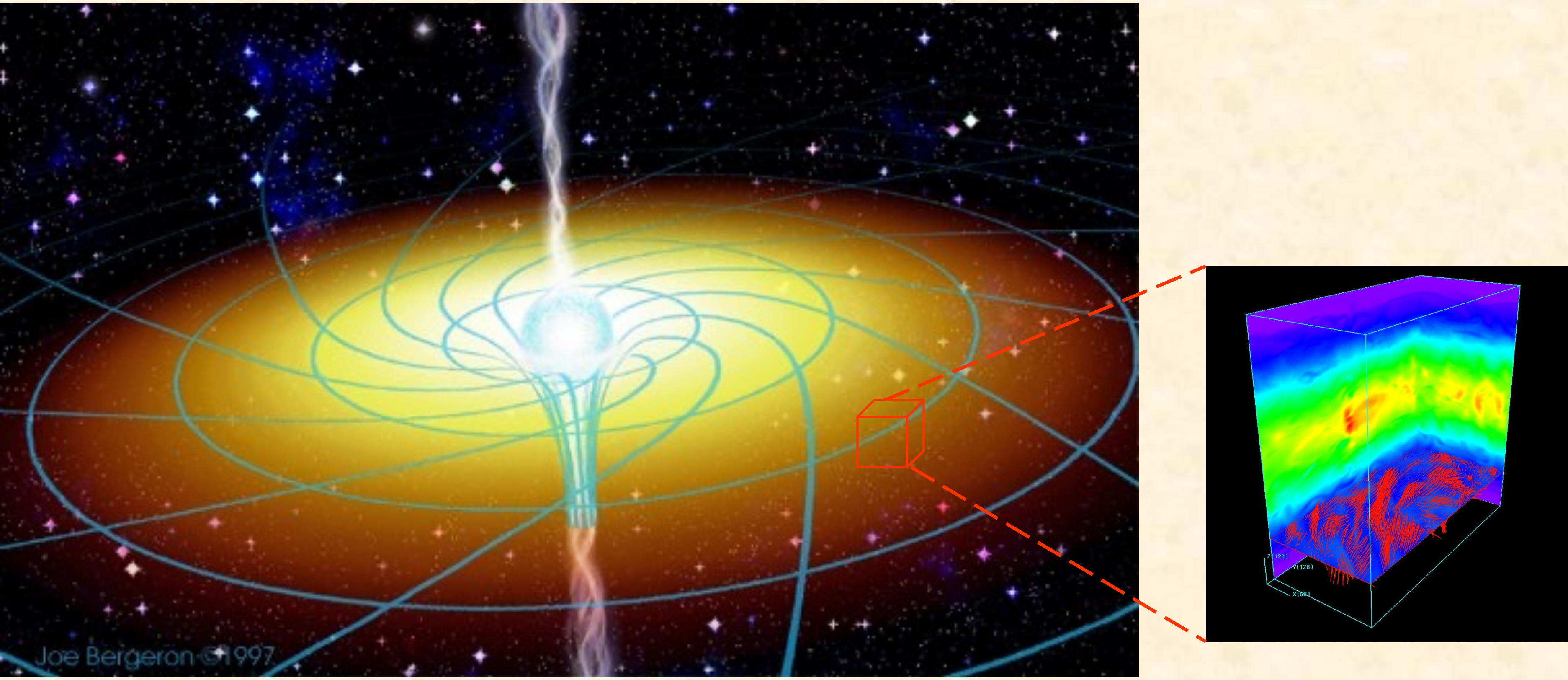


$k_Z V_A/\Omega$

Growth rate now peaks at specific k.

Analytic linear dispersion relation is a good code test.

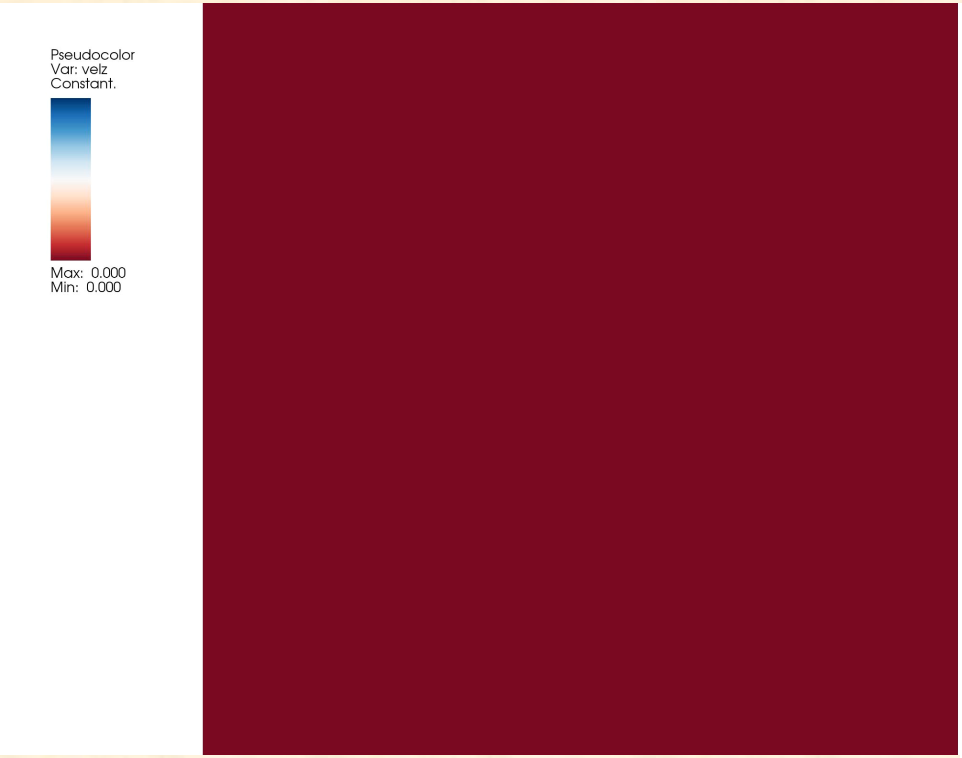
Saturation of the MRI has been studying in a small, local patch of a disk using the *shearing box*.



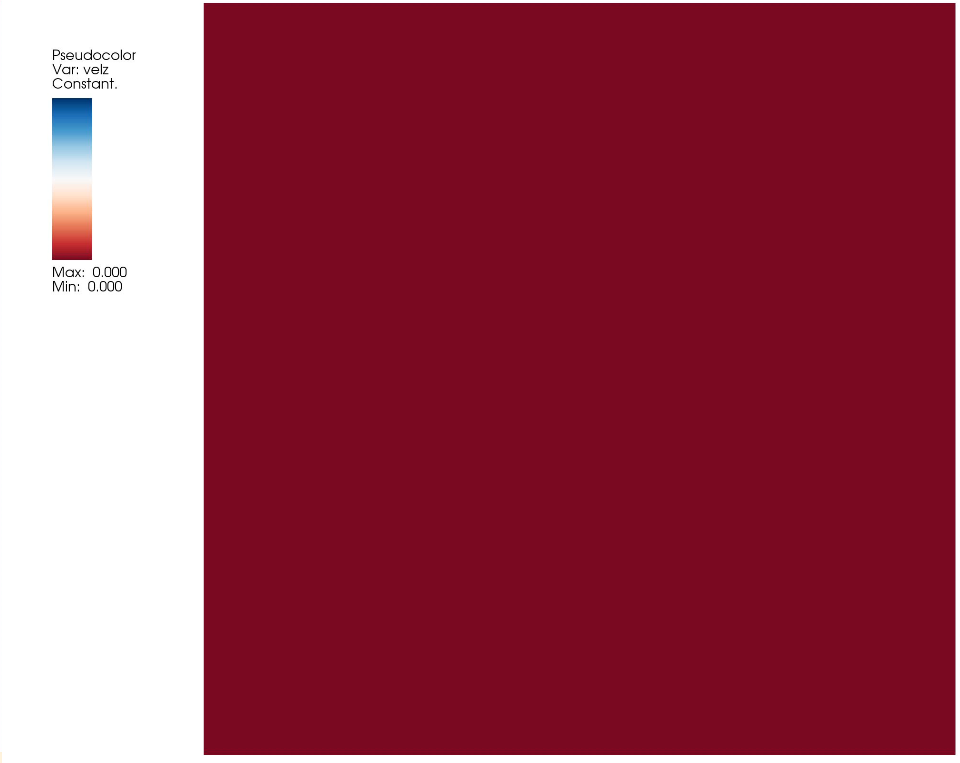
Solve equations in a frame co-rotating with flow at Keplerian frequency.
Requires special source terms for non-inertial frame, and special BCs in radius.

Nonlinear evolution of MRI in 2D

Net vertical flux



No-net vertical flux

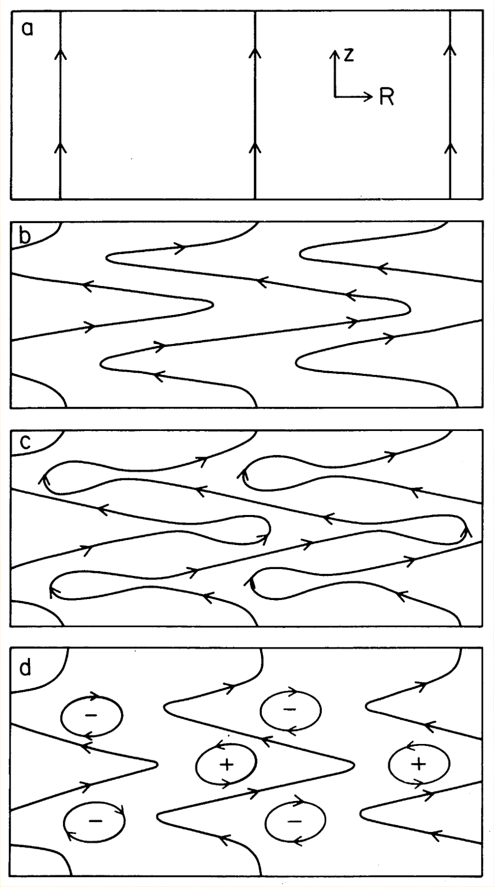


Images of $\delta V_\phi = V_\phi - V_{Keplerian}$

Net vertical flux in 2D: **channel modes**

Why do axisymmetric simulations with net flux blow-up?

- Axisymmetric modes of MRI are exact solutions to nonlinear MHD equations in local shearing box (Goodman & Xu 1994).
- Thus, these modes grow exponentially without bound, until parasitic instabilities (KH, reconnection) disrupt them.

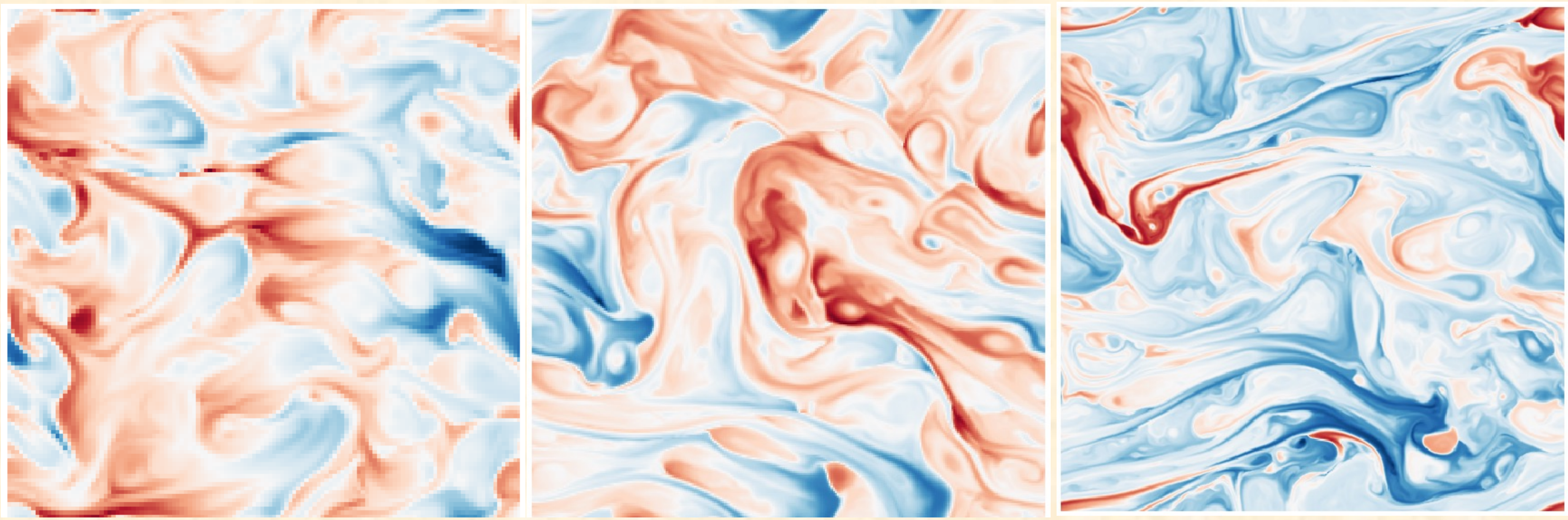


This does not happen in no-net flux case since varying field strength excites a range of unstable modes.

Schematic evolution of channel modes (Hawley & Balbus 1992)

No-net vertical flux in 2D: no convergence

Images of $\delta V_\phi = V_\phi - V_{Keplerian}$

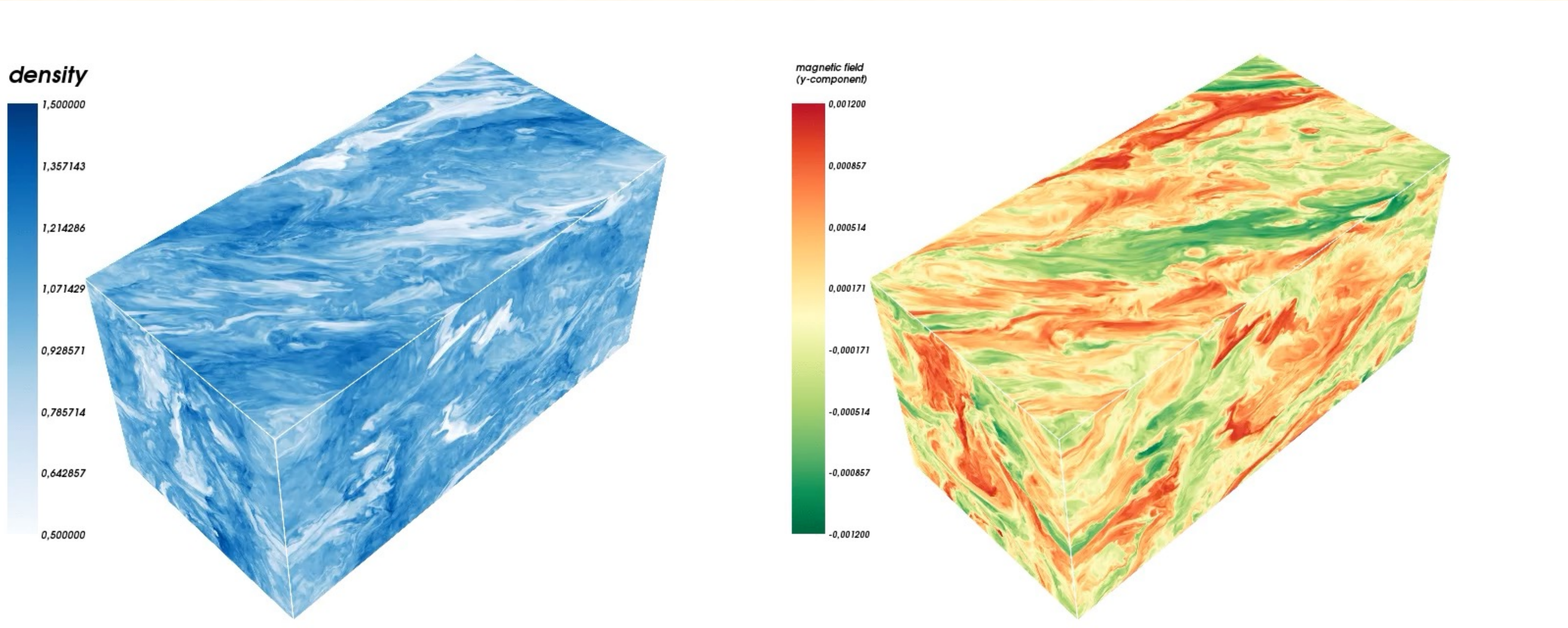


128^2 256^2 512^2

Without explicit dissipation, no-net-flux evolves into turbulence on ever smaller scales.

Nonlinear evolution of MRI in 3D with explicit dissipation

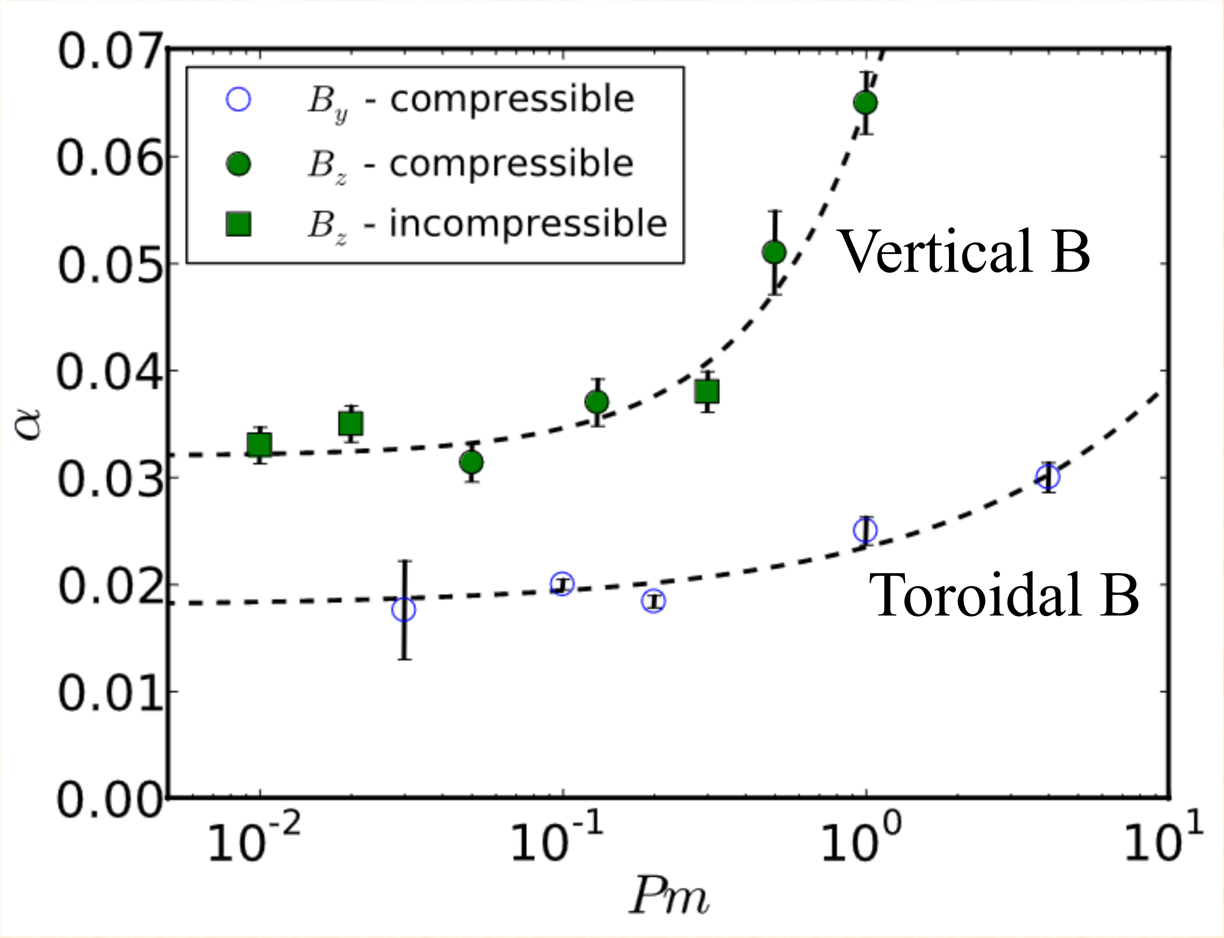
Converged solutions are obtained if both resistivity and viscosity are specified. Important dimensionless parameter is magnetic Prandtl number Pm = ν/η.



Re = 85000, Rm = 2600, Pm = 0.03, 800x1600x800 mesh.
(Meheut et al. 2015)

Pm dependence of angular momentum transport in MRI turbulence

Shear stress/pressure



Meheut et al. 2015

Summary

- One basic property of fluids is that they support linear (and in some cases nonlinear) propagating waves.

- For some equilibrium states, amplitude of linear waves grow exponentially in time, leading to instability.

- Numerical methods are important for studying the nonlinear regime of MHD instabilities.

- Generally, RTI, KHI, and MRI saturate as MHD turbulence in 3D. Explicit dissipation can be important in setting saturation amplitudes.