2023 NSF/GPAP School on Plasma Physics for Astrophysicists
Take-away questions and answers

1. What is the Debye length? How is an ideal plasma defined in terms of it, and what does this say about the importance of collective electrostatic interactions versus binary collisions?

The Debye length is given by

$$\lambda_{\mathrm{D}} \doteq \sqrt{\frac{T_{\mathrm{e}}}{4\pi e^2 n_{\mathrm{e}}}};$$

it measures the distance over which electrostatic effects due to charge separation persist in a plasma (or, equivalently, the characteristic e-folding distance beyond which charges are screened). Another acceptable answer, which takes into account the possible mobility of ions on plasma-frequency timescales, is

$$\lambda_{\mathrm{D}} \doteq \sqrt{\frac{T_{\mathrm{eff}}}{4\pi e^2 n_{\mathrm{e}}}}, \quad \text{where} \quad T_{\mathrm{eff}} \doteq \left(\frac{1}{T_{\mathrm{e}}} + \frac{Z_i n_{\mathrm{i}}}{n_{\mathrm{e}}}\frac{Z_{\mathrm{i}}}{T_{\mathrm{i}}}\right)^{-1}$$

and Z_i is the ion charge in units of e. A plasma is defined by $n_{\mathrm{e}}\lambda_{\mathrm{D}}^3 \ggg 1$ – i.e., the number of electrons per Debye sphere is very large. In this case, collective electrostatic interactions dominate over binary collisions.

2. Give a rough comparison ($\ll$, $<$, or $\sim$) of the ion Larmor radius ρ_{i}, the collisional mean free path λ_{mfp}, and the characteristic scale ℓ in the following plasmas: (*i*) the intracluster medium (cooling radius $\ell \sim 100$ kpc), (*ii*) the solar wind ($\ell \sim 1$ au), (*iii*) the warm phase of the interstellar medium ($\ell \sim 100$ pc), and (*iv*) the Galactic center (Bondi radius $\ell \sim 0.01$ pc). What is a typical degree of ionization in dense molecular clouds (i.e., at number densities $n_{\mathrm{H_2}} \gtrsim 10^3$ cm^{-3})?

(*i*) $\rho_{\mathrm{i}} \ll \lambda_{\mathrm{mfp}} < \ell$, (*ii*) $\rho_{\mathrm{i}} \ll \lambda_{\mathrm{mfp}} \sim \ell$, (*iii*) $\rho_{\mathrm{i}} \ll \lambda_{\mathrm{mfp}} \ll \ell$, (*iv*) $\rho_{\mathrm{i}} \ll \lambda_{\mathrm{mfp}} \sim \ell$. A typical degree of ionization in dense molecular clouds is $\lesssim 10^{-6}$.

3. Write down formulae for the $\boldsymbol{E}\times\boldsymbol{B}$ drift, the $\boldsymbol{\nabla}B$ drift, the curvature drift, and the polarization drift. In an ion-electron plasma, which of these drifts give rise to a current?

With $\boldsymbol{E}$ being the electric field and $\boldsymbol{B}$ being the magnetic field,

$$\boldsymbol{u}_{E\times B} = -\frac{\hat{\boldsymbol{b}}}{\Omega}\times\frac{q\boldsymbol{E}}{m} = \frac{c\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \quad \boldsymbol{u}_{\nabla B} = \frac{\hat{\boldsymbol{b}}}{\Omega}\times\frac{\mu\boldsymbol{\nabla}B}{m},$$

$$\boldsymbol{u}_{\mathrm{curv}} = \frac{\hat{\boldsymbol{b}}}{\Omega}\times\frac{m v_\parallel^2(\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla})\hat{\boldsymbol{b}}}{m}, \quad \boldsymbol{u}_{\mathrm{pol}} = \frac{1}{\Omega}\frac{c}{B}\frac{\partial\boldsymbol{E}_\perp}{\partial t},$$

where $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ is the unit vector in the magnetic-field direction, $\Omega \doteq qB/mc$ is the Larmor frequency, and $\mu \doteq m v_\perp^2/2B$ is the magnetic moment. The subscripts $\parallel$ and $\perp$ denote the directions parallel and perpendicular to the magnetic field. The $\boldsymbol{\nabla}B$, curvature, and polarization drifts give rise to currents, because the signs of these drift velocities are charge-dependent.

4. State all of the assumptions made in deriving the (nonrelativistic) ideal, single-fluid MHD equations.

Quasineutrality ($k\lambda_{\mathrm{D}} \ggg 1$), no displacement current ($v/c \ggg 1$), scalar pressure (i.e., Maxwell distribution to all orders: $\lambda_{\mathrm{mfp}} \ll L$), magnetized ($\rho_{\mathrm{i}} \ll L$), infinite conductivity, $u_{E\times B}/v_{\mathrm{th}} \sim 1$

5. Give the dispersion relation for a parallel-propagating Alfvén wave. Explain what physics drives the propagation of this wave.

The frequency of a parallel-propagating Alfvén wave is $\omega = \pm \boldsymbol{k} \cdot \boldsymbol{v}_{\mathrm{A}}$, where $\boldsymbol{k}$ is the wavevector and $\boldsymbol{v}_{\mathrm{A}} \doteq \boldsymbol{B}/\sqrt{4\pi\rho}$ is the Alfvén velocity. (Also acceptable is $\omega = \pm k_{\|} v_{\mathrm{A}}$, where $k_{\|}$ is the parallel component of the wavevector.) Alfvén waves result from a competition between magnetic tension, which provides the restoring force, and plasma inertia, which causes the magnetic-field perturbations to overshoot the equilibrium by virtue of flux-freezing. Oscillation ensues.

6. What are the physical set-ups that give rise to the Rayleigh–Taylor, Kelvin–Helmholtz, Jeans, Schwarzschild, and rotational instabilities? In each case, what is the free-energy source driving the instability and how does the growth rate scale with this free-energy source?

RTI: heavy fluid on top of light fluid in a gravitational field. Growth rate $\gamma = (|k| g \Delta)^{1/2}$, where k is the horizontal wavenumber, g is the gravitational acceleration, and $\Delta \doteq (\rho_{\mathrm{top}} - \rho_{\mathrm{bot}})/(\rho_{\mathrm{top}} + \rho_{\mathrm{bot}})$. Free energy is the gravitational potential energy. **KHI**: fast fluid neighboring slow fluid, with an inflection point in the shear flow. For a discontinuous x-jump in y-velocity from 0 to U, $\gamma \propto |k_x| U (1 \pm \mathrm{i})$. [Bonus, but not necessary: if the two fluids have differing densities, then the growth rate is accompanied by the factor $(\rho_{\mathrm{fast}} \rho_{\mathrm{slow}})^{1/2}/(\rho_{\mathrm{fast}} + \rho_{\mathrm{slow}})$.] Free energy is the shear flow. **Jeans**: self-gravitating sound wave with $|k| a < (4\pi G \rho)^{1/2}$. Growth rate $\gamma = (4\pi G\rho - k^2 a^2)^{1/2}$. **Schwarzschild**: convective instability when entropy increases in the direction of gravity. Growth rate $\gamma = |k_z/k| \sqrt{-N^2}$ where $N^2 = (g/\gamma)\, \mathrm{d} \ln P\rho^{-\gamma}/\mathrm{d}z < 0$. **Rayleigh**: outwardly decreasing angular momentum in a rotating fluid. Growth rate $\gamma = |k_z/k| \sqrt{-\kappa^2}$ where $\kappa^2 = R^{-3}\, \mathrm{d}(\Omega^2 R^4)/\mathrm{d}R < 0$ is the square of the epicyclic frequency and $\Omega = \Omega(R)$ is the angular velocity at cylindrical radius R. Generally doesn't happen, since $\kappa^2 > 0$ in most all astrophysical disks. **MRI**: outwardly decreasing angular *velocity* in a rotating fluid. Growth rate $\gamma \propto -\mathrm{d}\Omega/\mathrm{d}\ln R > 0$. Keplerian disks satisfy this.

7. Define the Lundquist number S. How does the reconnection rate in the Sweet–Parker model scale with S? Why is this a problem for explaining reconnection in most astrophysical plasmas?

The Lundquist number $S \doteq v_{\mathrm{A}} L/\eta$, where $v_{\mathrm{A}} = B/\sqrt{4\pi\rho}$ is the Alfvén speed, L is some characteristic (magnetic) lengthscale, and η is the magnetic resistivity. The reconnection rate in the Sweet-Parker model $\propto S^{-1/2}$. This is a problem for explaining reconnection in most astrophysical plasmas because most astrophysical plasmas have $S \ggg 1$. Physically, larger current density (i.e. a thin sheet) gives faster reconnection due to the larger free energy; but, in the Sweet–Parker model, this implies a narrow nozzle through which the exhaust escapes. The result is a slow reconnection rate.

8. State the assumptions used in Kolmogorov's theory of hydrodynamic turbulence. Use them to derive the energy spectrum $E(k)$, where $E = \int \mathrm{d}k\, E(k)$.

Assumptions are constant energy flux throughout the inertial range of the cascade from small to large wavenumbers (i.e., large to small scales), isotropy of the cascade ($\boldsymbol{k} \to k = 2\pi/\ell$), and locality of interactions. Then the energy spectrum is uniquely determined by dimensional analysis. The energy flux through scale ℓ is $\sim u_\ell^2/\tau_\ell \sim u_\ell^2/(\ell/u_\ell) = u_\ell^3/\ell$. This being constant implies $u_\ell \sim \ell^{1/3}$. The energy at scale ℓ is thus $u_\ell^2 \sim \ell^{2/3}$, and so $E(k) \sim u_k^2/k \sim k^{-5/3}$.

9. Describe the line of reasoning that leads to the Iroshnikov-Kraichnan ($k^{-3/2}$) and Goldreich-Sridhar ($k_\perp^{-5/3}$) spectra of MHD turbulence in the presence of a strong mean magnetic field. State all assumptions (namely, presence or absence of isotropy, relative magnitude of linear and nonlinear timescales). Why is the MHD turbulence spectrum not uniquely determined by purely dimensional considerations?

Define the Alfvén time $\tau_{\rm A} \doteq \ell_\| / v_{\rm A}$ and the strain (or "eddy") time $\tau_{\rm s} \doteq \ell / u_\ell$. **IK**: counterpropagating Alfvén wavepackets interact weakly ($\tau_{\rm A} \ll \tau_{\rm s}$), the relevant timescale being the Alfvén time. As in K41, isotropy of the cascade ($\ell_\| / \ell_\perp \sim 1$) and locality of interactions are assumed. **GS**: anisotropic cascade with comparable linear and non-linear frequencies, $k_\| v_{\rm A} \sim k_\perp u_\perp$ ("critical balance"). The principal difference with the hydrodynamic case is that the cascade time is no longer a dimensional inevitability, since two physical timescales are associated with each wave packet ($\tau_{\rm A}$ and $\tau_{\rm s}$). Put differently, there are three (rather than two) dimensionless combinations in the problem, and so further physics input is needed.

10. The magnetic moment of a particle $\mu \doteq m w_\perp^2 / 2B$, where $w_\perp$ is the component of the particle's peculiar ("thermal") velocity perpendicular to the magnetic field $\boldsymbol{B}$. State under what conditions μ is an adiabatic invariant.

μ is an adiabatic invariant if the particle sees a small change in the magnetic-field strength during a gyroperiod, whether that change is due to temporal or spatial variations in B (*viz.*, $|{\rm d} \ln B / {\rm d}t| \ll \Omega$).

11. What is Landau damping, and why is it not included in the MHD plasma description?

Landau damping is a resonant mechanism that collisionlessly damps electrostatic fluctuations, leading to the formation of small-scale structure in the velocity distribution function of the plasma particles. This small scale structure can eventually be smoothed by collisions, leading to irreversible heating. In any fluid description, the large rate of particle–particle collisions prevents the individual particles from resonating with the wave field.

12. Departures from an isotropic Maxwellian velocity distribution can lead to *kinetic instabilities*. What are three kinds of departures, and what are some linear instabilities that are generated by these departures?

Beams (i.e. well-separated populations with disparate velocities), bumps (i.e. overlapping or non-monotonically decreasing thermal distributions), and biases (i.e. anisotropies in the thermal motions of particles with respect to a particular direction). Beams lead to streaming instabilities; bumps to resonant instabilities; and biases to Weibel (and other Weibel-type) instabilities.

13. How does one obtain the Rankine–Hugoniot jump conditions for an MHD shock in steady state, and what do they provide? What is the asymptotic jump in density for a hydrodynamic, non-radiative shock when the Mach number $M \gg 1$? How does this density jump change in the presence of a magnetic field?

The Rankine–Hugoniot jump conditions provide the ratio of downstream to upstream values of density, flow velocity, pressure, and the components of the magnetic field normal and tangential to the shock front. They are obtained from the conservation of mass, momentum, and total energy, and the continuity of the shock-normal (-tangential) magnetic (electric) field, across the shock front. For a hydrodynamic shock with $M \gg 1$, $\rho_2/\rho_1 \approx (\gamma+1)/(\gamma-1)$ where γ is the ratio of specific heats (e.g., $\rho_2/\rho_1 \approx 4$ for $\gamma = 5/3$). This limit is unchanged in the presence of a magnetic field, except that we must require additionally that the Alfvén Mach number $M_{\rm A} \gg 1$ if $\boldsymbol{B} \perp \hat{\boldsymbol{n}}$.

14. What are cosmic rays and what is their observed energy spectrum?

Cosmic rays are high-energy charged particles (mostly hadrons) originating beyond Earth, with energies up to a few $\times 10^{20}$ eV. Those with energies up to $\sim 3\times 10^{15}$ eV are thought to be produced at supernova remnant shocks. The energy spectrum of cosmic rays is approximately a power law, with $\mathrm{d}N/\mathrm{d}E \propto E^{-2.7}$ over roughly 11 orders of magnitude in energy E (with a 'knee' and 'ankle').

15. What is the Stefan–Boltzmann Law and Planck's Law? When do they apply?

The Stefan–Boltzmann Law, σT^4, where σ is the Stefan–Boltzman constant, describes the power emitted from a blackbody. Note that is only dependent on temperature. It may be obtained from Planck's Law $B_\nu(\nu, T) = (2h\nu^3/c^2)(\mathrm{e}^{h\nu/k_\mathrm{B}T} - 1)^{-1}$ for the spectral density of electromagnetic radiation from a black body in thermal equilibrium at temperature T by integrating that formula over all frequencies and angles. A blackbody radiator emits at every frequency and radiates isotropically.

16. What is the Biermann battery and what is required for it to operate?

The Biermann battery uses misaligned density (n_e) and temperature (T_e) gradients in the electron fluid to generate (astrophysically weak) magnetic fields from zero initial conditions:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\left(\dots - \frac{\boldsymbol{\nabla} p_e}{en_e}\right) = \dots + \frac{c}{en_e^2}\boldsymbol{\nabla} T_e \times \boldsymbol{\nabla} n_e.$$

It is related to Kelvin's circulation theorem, but for the canonical vorticity corresponding to the canonical momentum $\boldsymbol{\wp}_e = m_e\boldsymbol{u} + e\boldsymbol{A}/c$, *viz.*, $\boldsymbol{\omega}_\mathrm{can} = m_e^{-1}\boldsymbol{\nabla}\times\boldsymbol{\wp}_e = \boldsymbol{\nabla}\times(\boldsymbol{u} + e\boldsymbol{A}/m_e c) = \boldsymbol{\omega} + e\boldsymbol{B}/m_e c$, where $\boldsymbol{A}$ is the vector potential. Physically, mis-aligned temperature and density gradients drive vorticity (see Kelvin's circulation theorem) and so, if the circulation corresponding to the canonical vorticity is conserved, then magnetic fields must arise. This is essentially equivalent to Lenz's law: loop currents driven by baroclinity cause magnetic fields.

Recommended textbooks and lecture notes for plasma physics:

- Alex Schekochihin, `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/KT/2015/KTLectureNotes.pdf`
- Mike Brown, `http://plasma.physics.swarthmore.edu/brownpapers/index.html`
- Matthew Kunz, `https://www.astro.princeton.edu/~kunz/Site/AST554/`
- Krall & Trivelpiece, *Principles of Plasma Physics*
- Dwight Nicholson, *Introduction to plasma theory*
- Hazeltine & Waelbroeck, *The Framework of Plasma Physics*
- Paul Bellan, *Fundamentals of Plasma Physics*
- Paul Drake, *High Energy Density Physics*

Recommended textbooks and lecture notes for plasma astrophysics and space physics:

- Matthew Kunz, `https://www.astro.princeton.edu/~kunz/Site/AST521/`
- Russell Kulsrud, *Plasma Physics for Astrophysics*
- Verscharen *et al.*, Living Review in Space Physics: `https://arxiv.org/pdf/1902.03448.pdf`

Recommend textbooks for fluid dynamics:

- D. J. Acheson, *Elementary Fluid Dynamics*
- Uriel Frisch, *Turbulence*