# MHD Relaxation

David N. Hosking

Lecture notes for the 2023 NSF/GPAP Summer School on plasma physics for astrophysicists.

## 1. Introduction

### 1.1 What is this subject about?

Let us imagine that we initialise an electrically conducting fluid (e.g., plasma) that is well described by the MHD equations in some non-trivial state. Then, we allow that configuration to relax, without external forcing, in the presence of some finite viscosity or resistivity that allows energy to decrease until a final state is reached. We shall be occupied with the question of ~~whether~~ how we can understand or even predict the relaxed state. (without having to solve the MHD equations!)

### 1.2 Why do we want to study this?

~~At first sight, this might seem like something of an abstract problem; after all, astrophysical systems are rarely~~

In some sense, the problem of MHD relaxation is a roundabout way of asking "which MHD equilibria are stable?" (relaxation → equilibrium state, and only a stable state can be the end point of this process). This might seem like something of an abstract question to an astrophysicist, as astrophysical systems are rarely in equilibrium; more often, energy is persistently ~~fee~~ resupplied so that the system is stirred dynamically in some way that makes any kind of self-organisation hard to maintain. Nonetheless, there remain many good reasons to study MHD relaxation:

1. Quasi-equilibria do exist: e.g. solar corona, primordial magnetic fields
2. Stability of MHD equilibria extremely important for lab experiments, e.g., SSX. (MHD relaxation theory originally developed in the context of magnetic-confinement fusion).
3. In tackling this problem we will introduce concepts that are of fundamental importance for understanding the origins of dynamical-scale magnetic fields in astrophysics (via a process known as the dynamo effect)

4. Most importantly, this problem will enable us to develop a tremendous ~~amount~~ amount of intuition for the physical nature of the MHD equations, and, particularly, for the role of the magnetic field's topology in constraining its evolution.

## 2. MHD Equilibria

As we have already noted, ~~free~~ relaxation will take us to some equilibrium state. It is therefore useful to build some intuition for the kinds of equilibria that are possible within MHD.

Static-equilibrium ($\underline{u} = 0$, $\frac{\partial}{\partial t} = 0$) solutions of the MHD equations must satisfy

$$-\underline{\nabla} p + \frac{\underline{j} \times \underline{B}}{c} = 0 \,, \quad \underline{j} = \frac{c}{4\pi} \underline{\nabla} \times \underline{B} \,, \quad \underline{\nabla} \cdot \underline{B} = 0 \,.$$

you have already seen that this term (the Lorentz force) can be decomposed into magnetic-tension force + magnetic-pressure force, i.e.

$$\frac{\underline{j} \times \underline{B}}{c} = \underbrace{\frac{B^2}{4\pi} (\hat{\underline{b}} \cdot \underline{\nabla} \hat{\underline{b}})}_{\text{mag. tension}} \underbrace{- (\underline{\underline{I}} - \hat{\underline{b}}\hat{\underline{b}}) \cdot \underline{\nabla} \frac{B^2}{8\pi}}_{\text{mag. pressure}}$$

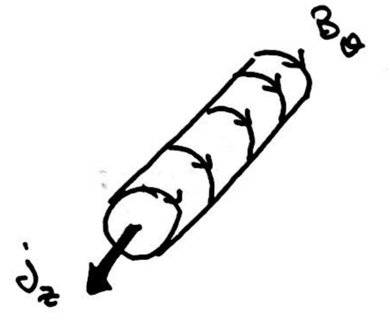
MHD equilibria therefore boil down to some balance of magnetic tension with magnetic and thermal pressure gradients.

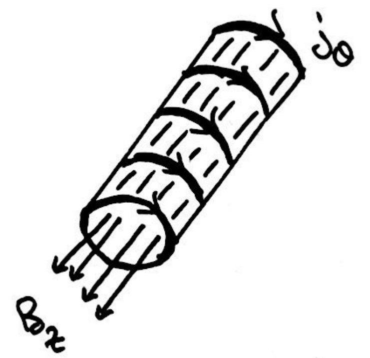
In cylindrical coordinates $(r, \theta, z)$ with $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0$, these equations take a simple form:

$$\underbrace{\frac{\partial}{\partial r}\left( p + \frac{B^2}{8\pi} \right)}_{\text{total pressure gradient}} = \underbrace{- \frac{B_\theta^2}{4\pi r}}_{\text{tension force}} .$$

(Exercise: derive this!)

Two solutions are the so-called "z-pinch" and the "$\theta$-pinch"




$\underline{B_\theta = 0}$

$\underline{z\text{-pinch}}$: magnetic tension balances magnetic + thermal pressure

$\underline{\theta\text{-pinch}}$: no magnetic tension, magnetic and thermal pressures in balance.

(Exercise: derive equations for these!)

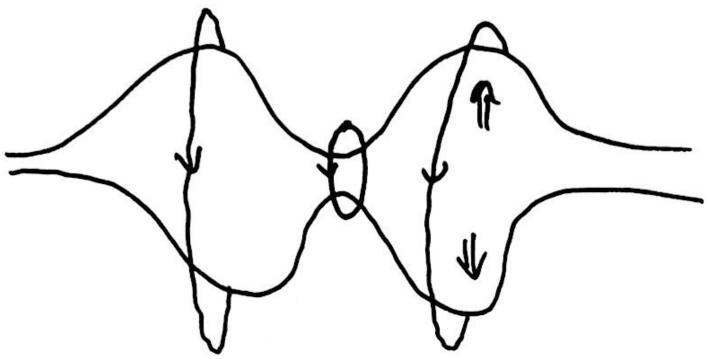
There is another class of equilibria, called force-free equilibria, for which the Lorentz force vanishes, i.e.

$$\underline{j} \times \underline{B} = 0 \Rightarrow \underline{j} \parallel \underline{B} \Rightarrow \frac{4\pi}{c}\underline{j} = \boxed{\underline{\nabla} \times \underline{B} = \alpha(\underline{x})\,\underline{B}}$$

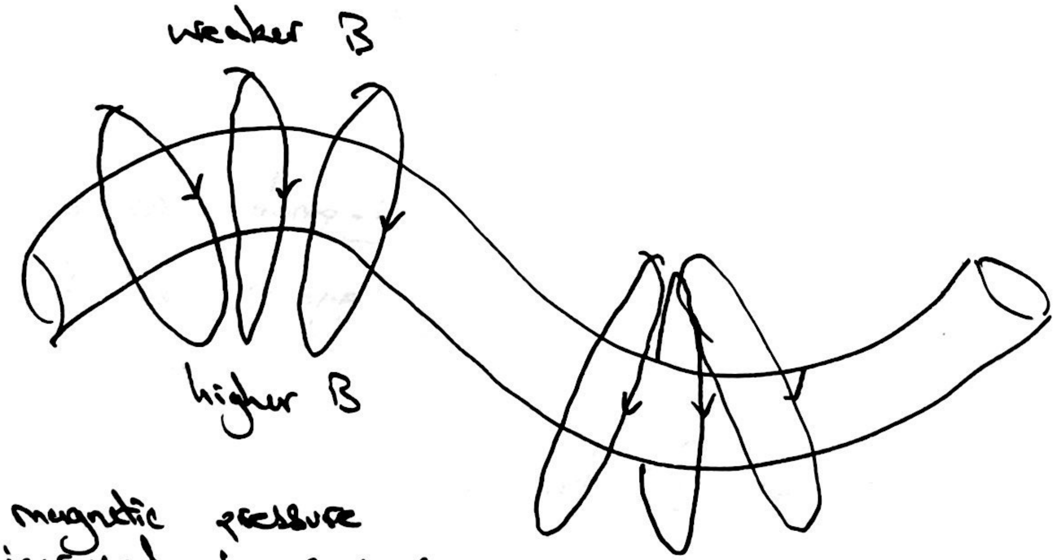
$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow \underline{B} \cdot \underline{\nabla}\alpha = 0 \Rightarrow \alpha$ is constant on magnetic field lines.

## Aside: instabilities of the z-pinch:

Sausage instability: Compressible ($\underline{\nabla} \cdot \underline{u} \neq 0$)



Kink instability Incompressible ($\underline{\nabla} \cdot \underline{u} = 0$)




→ magnetic pressure increased in concave part

→ gradient of mag. pressure, drives further kink.

iv) In environments where $\beta \equiv p/(B^2/8\pi) \ll 1$, so thermal pressure is negligible compared to magnetic, force-free equilibria are the only possibility. Force-free equilibria satisfying $\underline{\nabla} \times \underline{B} = \alpha(\underline{x})\, \underline{B}$ are often used to model magnetic fields in the solar corona.

The special case of $\alpha = \text{const}$ (everywhere) is called a <u>linear force-free field</u>.

So, there ~~is~~ are, it seems, many different possible MHD equilibria. This brings us back to our original problem: which of ~~these~~ these equilibria are "most natural", in the sense that they will be reached by ~~some~~ some non-trivial relaxing configuration?

I have already mentioned that we are allowing energy to be dissipated by viscosity and/or resistivity during the relaxation (otherwise there would be no hope of arriving at a static final state). In some sense, therefore, the final state is one of minimum magnetic energy:

$$\int dV \, \frac{B^2}{8\pi} \rightarrow \min .$$

Without any further constraints, the solution is obvious,

$$\underline{B} = 0.$$

Not very interesting!

<u>But</u> there <u>are</u> further constraints.

Recall Alfvén's theorem from yesterday's lecture on MHD, which states that magnetic field is frozen into the fluid flow: as the fluid moves, the magnetic field is brought along with it. If we imagine our relaxing field to be a tangled mess, we see that this imposes constraints of a <u>topological</u> nature: linkages, knots etc. cannot be easily undone.

We can turn this intuition into a quantitative theory once we discover that MHD actually has a so-far unmentioned invariant that is, in a sense, "better conserved" than energy.

# 3. Magnetic helicity

## 3.1 Definition

$$H = \int_V dV \, \underline{A} \cdot \underline{B}$$

where $\underline{A}$ is the magnetic vector potential, $\underline{\nabla} \times \underline{A} = \underline{B}$.

## 3.2 Helicity is ~~independent of~~ well defined

$\underline{A}$ is not unique: a gauge transformation

$$\underline{A} \to \underline{A} + \underline{\nabla} \chi ,$$

with $\chi$ an arbitrary scalar function, leaves $\underline{B}$ unchanged.

Under this gauge transformation,

$$H \to H + \int_V dV \, \underbrace{\underline{B} \cdot \underline{\nabla} \chi}_{\substack{\shortparallel \\ \underline{\nabla} \cdot (\underline{B} \chi)}} = H + \int_{\partial V} d\underline{S} \cdot \underline{B} \chi = H$$

provided $\underline{B}$ at the boundary is parallel to the boundary: no magnetic field sticks out of $V$.

## 3.3 Helicity is conserved

The MHD induction equation reads

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \left( \underline{u} \times \underline{B} - \eta \, \underline{\nabla} \times \underline{B} \right)$$

Uncurling:

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u}\times\mathbf{B} - \eta\,\nabla\times\mathbf{B} + \nabla\chi$$

From these two equations, we can compute the evolution of $\mathbf{A}\cdot\mathbf{B}$:

$$\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B} = \left(\cancel{\mathbf{u}\times\mathbf{B}} - \eta\,\nabla\times\mathbf{B} + \nabla\chi\right)\cdot\mathbf{B} + \underbrace{\overset{a}{\mathbf{A}}\cdot\Big[\nabla\times\overbrace{\left(\mathbf{u}\times\mathbf{B} - \eta\,\nabla\times\mathbf{B}\right)}^{b}\Big]}_{\text{use } \nabla\cdot(\mathbf{a}\times\mathbf{b}) = (\nabla\times\mathbf{a})\cdot\mathbf{b} - (\nabla\times\mathbf{b})\cdot\mathbf{a}}$$

$$= -\eta\,\mathbf{B}\cdot(\nabla\times\mathbf{B}) + \nabla\cdot(\mathbf{B}\chi)$$

$$- \nabla\cdot\left[\mathbf{A}\times\left(\mathbf{u}\times\mathbf{B} - \eta\,\nabla\times\mathbf{B}\right)\right] + \left(\cancel{\mathbf{u}\times\mathbf{B}} - \eta\,\nabla\times\mathbf{B}\right)\cdot\underbrace{(\nabla\times\mathbf{A})}_{\mathbf{B}}$$

$$= \nabla\cdot\left[\mathbf{B}\chi - \mathbf{u}\,\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\,\mathbf{A}\cdot\mathbf{u} + \eta\,\mathbf{A}\times(\nabla\times\mathbf{B})\right] - 2\eta\,\mathbf{B}\cdot(\nabla\times\mathbf{B})$$

So

$$\frac{dH}{dt} = \int_V dV\,\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B} = \int_{\partial V} d\mathbf{S}\cdot\left[\mathbf{B}\chi - \mathbf{u}\,\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\,\mathbf{A}\cdot\mathbf{u} + \eta\,\mathbf{A}\times(\nabla\times\mathbf{B})\right]$$

$$- 2\eta\int_V dV\,\mathbf{B}\cdot(\nabla\times\mathbf{B})$$

If we choose $V$ to be such that $\mathbf{B}$ vanishes on $\partial V$ and $\mathbf{u}$ is parallel to ~~&~~ $\partial V$, then

$$\frac{dH}{dt} = -2\eta \int dV \, \underline{B} \cdot (\underline{\nabla} \times \underline{B}). \qquad (*)$$

We conclude that magnetic helicity is conserved in ideal MHD.

More importantly, though, H is better conserved than energy as $\eta \to 0$. To see this, note that

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

$$\Rightarrow \frac{\partial}{\partial t} \int dV \left(\frac{B^2}{8\pi}\right) = \begin{pmatrix} \text{energy exchanges} \\ + \text{fluxes} \end{pmatrix} - \frac{\eta}{4\pi} \int dV \, |\underline{\nabla} \times \underline{B}|^2 .$$

If energy decay happens at a finite rate as $\eta \to 0$, this must be because $\underline{B}$ develops ~~scales~~ fine scales, such that $\nabla \sim \eta^{-1/2}$. But, $(*)$ then tells us that

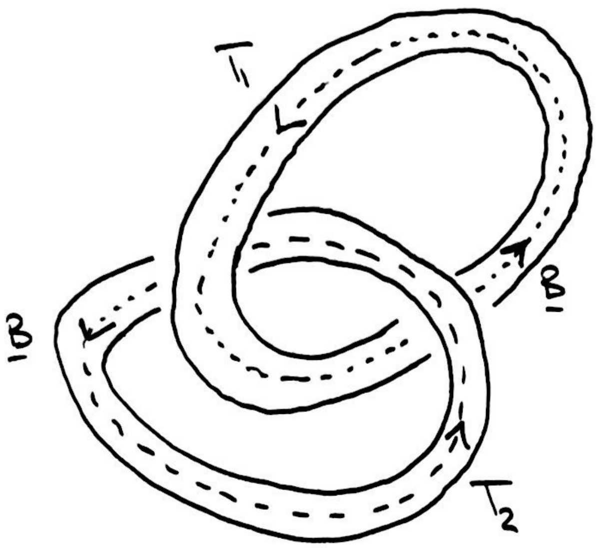
$$\frac{dH}{dt} = O(\eta^{1/2}) \to 0 \quad \text{as} \quad \eta \to 0.$$

Therefore, as $\eta \to 0$, magnetic helicity is conserved even as magnetic energy decays.

The conservation of H is the constraint subject to which we will have to minimise energy!

Before we do that, let us discover what magnetic helicity means, physically.

## 3.4 Helicity is a topological invariant




Consider the two linked flux tubes shown. The helicity of $T_1$ is

$$H_1 = \int_{T_1} dV \, \underline{A}\cdot\underline{B} = \int_{T_1} \underbrace{d\underline{l}}_{\hat{\underline{b}}\,dl} \cdot \underbrace{d\underline{S}}_{\hat{\underline{b}}\,dS} \; \underline{A}\cdot\hat{\underline{b}}\,B$$

$$= \int_{T_1} \underline{A}\cdot\hat{\underline{b}}\,dl \;\; B\hat{\underline{b}}\cdot\hat{\underline{b}}\,dS$$

$$= \int_{T_1} \underline{A}\cdot d\underline{l} \;\; \underline{B}\cdot d\underline{S} = \Phi_1 \underbrace{\int_{T_1} \underline{A}\cdot d\underline{l}}_{\int_{S_1} \underline{B}\cdot d\underline{S} \,=\, \Phi_{\text{hole in } T_1} \,=\, \Phi_2} = \Phi_1 \Phi_2$$

In general, for many linked tubes,

$$H_i = \Phi_i \, \Phi_{\text{hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}$$

defined overleaf

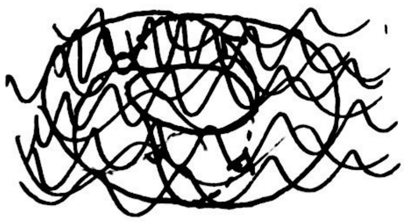
and the total helicity of the whole system is

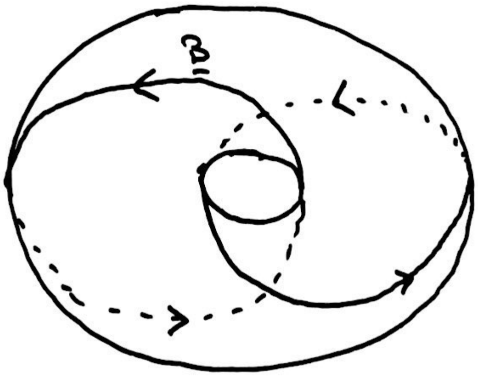
$$H = \sum_i \Phi_i \sum_j \Phi_j N_{ij} = \sum_{ij} \Phi_i \Phi_j N_{ij} .$$

the number of times that tube $j$ passes through the hole in tube $i$

" $H$ is the number of linkages of the flux tubes weighted by the field strength in them" .

Given this topological interpretation, it is natural to wonder what it means to say that helicity is better conserved than magnetic energy even in the presence of finite diffusivity, when field-line topology is no longer invariant. The answer is that the breaking and "re-connection" of field lines (a process about which you will learn much more later in the school) that is allowed at finite ~~res~~ ? also conserves magnetic helicity, even through topology changes. This happens because reconnecting linked field lines and up twisted, and twists are associated with helicity:






This figure designed to show that field lines in a twisted toroidal configuration link each other $\Rightarrow$ helicity.

## 4. J. B. Taylor Relaxation

We now know how to find relaxed magnetic states! We minimise energy subject to constant helicity. But how do we minimise things subject to constraints?

### 4.1 Mathematical interlude: Lagrange multipliers

Suppose we have some function $F(x_1, \dots, x_n)$ that we want to extremise subject to the constraint $G(x_1, \dots, x_n) = 0$.

At the extremum, $dF = \dfrac{\partial F}{\partial x_1} dx_1 + \dots + \dfrac{\partial F}{\partial x_n} dx_n = 0$

ordinarily, we would set $\dfrac{\partial F}{\partial x_i} = 0$, but these are not independent

we also have $$dG = \frac{\partial G}{\partial x_1} dx_1 + \dots + \frac{\partial G}{\partial x_n} dx_n = 0$$

solve for $dx_1$: $$dx_1 = -\left( \frac{\partial G/\partial x_2}{\partial G/\partial x_1} dx_2 + \dots \frac{\partial G/\partial x_n}{\partial G/\partial x_1} dx_n \right)$$

subs into $dF$ eqn:

$$dF = \left( \frac{\partial F}{\partial x_2} - \underbrace{\frac{\partial F/\partial x_1}{\partial G/\partial x_1}}_{\equiv \alpha} \frac{\partial G}{\partial x_2} \right) dx_2 + \dots + \left( \frac{\partial F}{\partial x_n} - \underbrace{\frac{\partial F/\partial x_1}{\partial G/\partial x_1}}_{\equiv \alpha} \frac{\partial G}{\partial x_n} \right) dx_n$$

now independent

so, we now have to solve $\dfrac{\partial F}{\partial x_i} - \alpha \dfrac{\partial G}{\partial x_i} = 0$ ($i=1$ holds by definition of $\alpha$)

and $G = 0$.

But, these are precisely the Eqns. we would need to solve to extremise $F - \alpha G$ unconstrainedly w.r.t. $x_1, \dots, x_n, \alpha$, as then

$$d(F - \alpha G) = \sum_i \left( \frac{\partial F}{\partial x_i} - \alpha \frac{\partial G}{\partial x_i} \right) - G\, d\alpha = 0.$$

This, then, is the method of Lagrange multipliers: take the function you want to ~~maximise~~ extremise, subtract $\alpha \times$ constraint, and then extremise unconstrainedly.

## 4.2 Example: Dido's problem

According to legend, the Phoenician queen Dido founded the ancient city of Carthage in modern-day Tunisia in 814 BC after fleeing her scheming brother who had killed her husband. The story goes that she asked the locals for as much land

as could be bounded by a bull's hide. From Virgil's 
Aeneid:

"They came to this spot, where today you can behold
the mighty
Battlements and the rising citadel of New Carthage,
And purchased a site, which was named 'Bull's Hide'
after the bargain
By which they should get as much land as they could
enclose with a bull's hide".

According to the story, Dido cut the hide into a long thin strip and bounded the maximum possible area with this. The shape that she marked out was a <u>circle</u> because a circle has the maximum area for a given perimeter. We can prove this with our new technique of Lagrange multipliers:

The neatest way to do this is to use the formula

area $\rightarrow A = \frac{1}{2} \oint dt\,(x\dot{y} - y\dot{x})$, which follows from

Green's theorem, $\int_{\partial D} (f\,dx + g\,dy) = \int_D \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$,

with $f = -\frac{y}{2}$, $g = \frac{x}{2}$. (Green's theorem is a 2D statement of the familiar Stokes' theorem).

Our constraint is $\oint dt\, \sqrt{\dot{x}^2 + \dot{y}^2} = \text{const} = C$

$\Rightarrow$ we ~~extremise unconstrainedly~~ seek

$$\delta\left[\int dt\left(\tfrac{1}{2}x\dot{y} - \tfrac{1}{2}y\dot{x} - \lambda\sqrt{\dot{x}^2+\dot{y}^2}\right) + \lambda C\right] = 0 .$$

Lagrange multiplier

~~Terms~~ The terms in which $x$ is varied are

$$\int dt\left(\tfrac{1}{2}\dot{y}\,\delta x - \left[\tfrac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right]\delta\dot{x}\right) + \dots$$

$$= \int dt\,\delta x\,\underbrace{\left[\tfrac{1}{2}\dot{y} + \frac{d}{dt}\left(\tfrac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)\right]}_{=0 \text{ since } \delta x \text{ is arbitrary.}} + \dots$$

by parts

Integrate: $\quad y + \dfrac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = b$, a constant.

Likewise, varying the $y$-related terms, we get

$$x - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = a, \quad \text{another constant.}$$

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = \lambda^2$$

which is indeed a circle (we can get $\lambda$ by using the constraint equation)

## 4.3 MHD Relaxation

Lagrange multipliers in hand, let us now finally work out which MHD equilibria have the smallest energy for fixed magnetic helicity. We ~~[illegible]~~ ~~[illegible]~~: seek

$$\delta\left[\int_V dV\left(\underset{(1)}{B^2}-\alpha\underset{(2)}{\underline{A}\cdot\underline{B}}\right)+\alpha H\right]=0$$

(1): $$\delta\int_V dV\, B^2 = 2\int dV\,\underline{B}\cdot\delta\underline{B} = 2\int dV\,\underline{B}\cdot(\underline{\nabla}\times\delta\underline{A})$$

$$= 2\int dV\left[-\underline{\nabla}\cdot(\underline{B}\times\delta\underline{A})+(\underline{\nabla}\times\underline{B})\cdot\delta\underline{A}\right]$$

$$= -2\int_{\partial V} d\underline{S}\cdot(\underline{B}\times\delta\underline{A})+2\int_V dV\,(\underline{\nabla}\times\underline{B})\cdot\delta\underline{A}$$

(2): $$\delta\int_V dV\,\underline{A}\cdot\underline{B} = \int_V dV\left[\underline{B}\cdot\delta\underline{A}+\underbrace{\underline{A}\cdot\delta\underline{B}}_{=\,\underline{A}\cdot(\underline{\nabla}\times\delta\underline{A})}\right]$$

$$= \int_V dV\left[\underline{B}\cdot\delta\underline{A}-\underline{\nabla}\cdot(\underline{A}\times\delta\underline{A})+\underbrace{(\underline{\nabla}\times\underline{A})}_{\underline{B}}\cdot\delta\underline{A}\right]$$

$$= -\int_{\partial V} d\underline{S}\cdot(\underline{A}\times\delta\underline{A})+2\int_V dV\,\underline{B}\cdot\delta\underline{A}$$

The surface integrals in each of these final expressions vanish if the variations are made in a way that respects flux freezing:

displacement field, small

From $$\frac{\partial \delta \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B}) = \underline{\nabla} \times \left( \frac{\partial \underline{\xi}}{\partial t} \times \underline{B} \right)$$

we have $$\delta \underline{B} = \underline{\nabla} \times (\underline{\xi} \times \underline{B})$$
$$\delta \underline{A} = \underline{\xi} \times \underline{B}$$

so $$\underline{B} \times \delta \underline{A} = B^2 \underline{\xi} - \underline{B} \cdot \underline{\xi}\, \underline{B}$$
$$\underline{A} \times \delta \underline{A} = \underline{A} \cdot \underline{B}\, \underline{\xi} - \underline{A} \cdot \underline{\xi}\, \underline{B}$$

are both $\perp$ $d\underline{S}$ if $\underline{\xi}, \underline{B}$ are parallel to the boundary.

We are left with

$$\delta \left[ \int dV \left( B^2 - \alpha \underline{A} \cdot \underline{B} \right) \right] = 2 \int dV \underbrace{\left( \underline{\nabla} \times \underline{B} - \alpha \underline{B} \right)}_{\text{must } = 0} \cdot \delta \underline{A} = 0 .$$

$$\Rightarrow \quad \underline{\underline{\nabla} \times \underline{B} = \alpha \underline{B}}$$ linear force-free field.

or, equivalently, taking the curl of each side,

$$\nabla^2 \underline{B} = -\alpha^2 \underline{B} .$$

To find relaxed, force-free equilibria:

1. Solve $\nabla^2 \underline{B} = -\alpha^2 \underline{B}$ for $\underline{B} = \underline{B}(\alpha)$, parametrically dependent on $\alpha$

2. Calculate $H(\alpha)$ via

$$H(\alpha) = \int dV \, \underline{A}.\underline{B} = \frac{1}{\alpha} \int dV \, B^2$$

3. Set $H(\alpha) = H_0$, where $H_0$ is the initial value of the helicity, thus find $\alpha = \alpha(H_0)$

4. Complete the solution by using this $\alpha$ in $\underline{B} = \underline{B}(\alpha)$

If this procedure returns multiple solutions, we pick the one with smallest energy: that is the one that is guaranteed to be stable.