Mathematical Preliminaries

Vector-algebra relations:

$$\underline{A}\cdot(\underline{B}\times\underline{C}) = \underline{B}\cdot(\underline{C}\times\underline{A}) = \underline{C}\cdot(\underline{A}\times\underline{B})$$

$$\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$$

$$\underline{\nabla}\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot(\underline{\nabla}\times\underline{A}) - \underline{A}\cdot(\underline{\nabla}\times\underline{B})$$

$$\underline{\nabla}\times(\underline{A}\times\underline{B}) = (\underline{B}\cdot\underline{\nabla})\underline{A} - (\underline{A}\cdot\underline{\nabla})\underline{B} - \underline{B}(\underline{\nabla}\cdot\underline{A}) + \underline{A}(\underline{\nabla}\cdot\underline{B})$$

$$\underline{A}\times(\underline{\nabla}\times\underline{B}) + \underline{B}\times(\underline{\nabla}\times\underline{A}) = \underline{\nabla}(\underline{A}\cdot\underline{B}) - (\underline{A}\cdot\underline{\nabla})\underline{B} - (\underline{B}\cdot\underline{\nabla})\underline{A}$$

...

these can be derived from the relation

$$\varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$$

Levi-Civita tensor

Kronecker delta

Integral formulae:

Divergence theorem: $\displaystyle\int_V dV\,\underline{\nabla}\cdot\underline{F} = \int_{\partial V} d\underline{S}\cdot\underline{F}$

useful corollary: $\displaystyle\int_V dV\,\underline{\nabla}\phi = \int_{\partial V} d\underline{S}\,\phi$

Stokes' theorem: $\displaystyle\int_S d\underline{S}\cdot\underline{\nabla}\times\underline{F} = \oint_{\partial S} d\underline{l}\cdot\underline{F}$

Fundamentals of Fluid Mechanics

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Lecture notes for the 2023 NSF/GPAP Summer School on plasma physics for astrophysicists.

1. Basic ideas

Fluid mechanics is based on the "continuum hypothesis": fluids are indivisible, continuous objects. This means that we can speak of such things as the density $\rho(\underline{x}, t)$ ~~at~~ and velocity $\underline{u}(\underline{x}, t)$ at position $\underline{x}$ (and time t). In reality, we know that the hypothesis is false - fluids are made of atoms - but we also know from experience that this fact is not very important for understanding the large-scale properties of many fluids.

As well as discreteness, we ignore thermal motions. $\underline{u}(\underline{x}, t)$ will be a coarse-grained quantity.

How should we think about fields like $\rho(\underline{x}, t)$ and $\underline{u}(\underline{x}, t)$?

Two ways: 1. as quantities defined at fixed $\underline{x}$: "Eulerian"
(our equations will be written using the Eulerian description)

2. as quantities defined for "fluid particles" (not real molecules!) following trajectories $\underline{x}(t)$. Often useful for intuition.

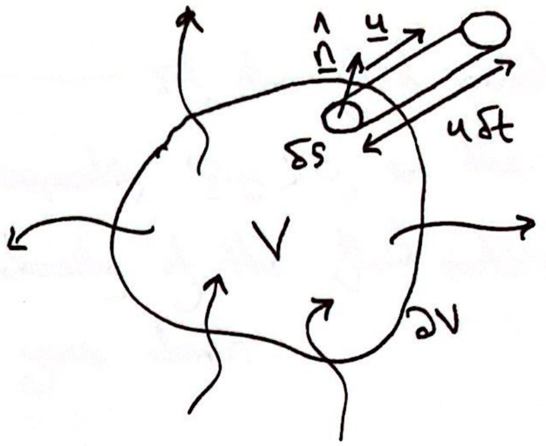
2. The equations of fluid mechanics

We shall derive these from familiar conservation laws: mass, momentum, energy.

2.1 Conservation of mass → "the continuity equation"

The mass inside some *fixed* volume V is

$$M = \int_V dV \rho$$



M can vary with time because fluid can flow in and out of V.

The flux of mass through a small part of the surface of V is $\delta M = \rho \underline{u} \cdot \hat{\underline{n}} \, \delta S \, \delta t$. Integrating over the surface, we find that

$$\frac{d}{dt} \int_V dV \rho = \int_V dV \frac{\partial \rho}{\partial t} = - \int_{\partial V} d\underline{S} \cdot \rho \underline{u} + S$$

$$\Rightarrow \int_V dV \frac{\partial \rho}{\partial t} = - \int_V dV \, \underline{\nabla} \cdot (\rho \underline{u}) + S \qquad (*)$$

where S represents possible sources or sinks of mass (e.g. inflow / outflow at boundary, exotic processes like pair production / annihilation). Neglecting S, and noting that $(*)$ must hold for any volume V, we find that

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0.}$$

This is called the continuity equation – it is the differential form of mass conservation. It can be rewritten

$$\underbrace{\frac{\partial \rho}{\partial t} + \underline{u} \cdot \underline{\nabla} \rho}_{} = - \rho \underline{\nabla} \cdot \underline{u}$$

rate of change of density at fixed point in space (Eulerian)

If the fluid particle expands, $\underline{\nabla} \cdot \underline{u} > 0$, so the density of the fluid particle goes down.

If $\rho = \rho(\underline{x}(t), t)$, i.e. we think of the density as being defined for a moving fluid particle, then

$$\frac{d}{dt} \rho(\underline{x}(t), t) = \frac{\partial \rho}{\partial t} + \frac{d\underline{x}(t)}{dt} \cdot \frac{\partial \rho}{\partial \underline{x}}$$

$$= \frac{\partial \rho}{\partial t} + \underline{u} \cdot \underline{\nabla} \rho$$

You will often see this written $\frac{D}{Dt}$, and referred to as the Lagrangian derivative (or convective, or comoving)

2.2 Newton's second law: "the momentum equation"

Newton's second law tells us that momentum is conserved, up to sources and sinks that we call "forces". We already have the mathematical machinery required to formalise this statement for fluids. Eq. (**) tells us that

$$\int_V dV \, \frac{\partial \varphi}{\partial t} = - \int_V dV \, \underline{\nabla} \cdot (\varphi \underline{u}) + S \qquad (+)$$

for φ the density of any conserved quantity possessed by the fluid. For conserved quantities that are vectors, we may promote $\varphi \to \varphi_i = \rho u_i$, for example. This is the momentum density. Eq. (+) becomes

$$\int_V dV \, \frac{\partial}{\partial t} (\rho u_i) = - \int_V dV \, \partial_j (\rho u_i u_j) + f_i ,$$

where f_i represents the (i-th component of) the forces acting on the fluid inside the volume V. For example,

the force due to pressure: $\displaystyle - \int_{\partial V} d\underline{S} \, p = - \int_V dV \, \underline{\nabla} p ,$

viscosity: $\displaystyle \int_{\partial V} dS_j \, \sigma_{ij} = \int_V dV \, \partial_j \sigma_{ij}$

the viscous stress tensor. More later.

Again, the volume V is arbitrary, so it must be the case that

$$\underbrace{\frac{\partial}{\partial t}(\rho u_i) + \partial_j(\rho u_i u_j)} = -\partial_i p + \partial_j \sigma_{ij} + f_i$$

in a slight abuse of notation, f will now be any other ~~forces~~ forces, e.g. gravity.

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \partial_j u_i + u_i \underbrace{\left[\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j)\right]}_{=0,\ \text{continuity eqn.}}$$

$$\Rightarrow \boxed{\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}\right) = -\underline{\nabla} p + \underline{\nabla}\cdot\underline{\underline{\sigma}} + \underline{f}}$$

Note the reappearance of the Lagrangian derivative on the left-hand side! $\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}$ is the acceleration of a fluid parcel, not $\frac{\partial \underline{u}}{\partial t}$. Thus, this equation is the fluid version of "$f = ma$".

Important note: $[\underline{u}\cdot\underline{\nabla}\,\underline{u}]_i = u_j \partial_j u_i$ only in cartesian coordinates. In curvilinear coordinates (cylindrical or spherical) one has to be careful because the ∂_j also hits the unit vectors. See lecture by M. Kunz

2.3 Conservation of energy

Let us denote by e the internal (thermal) energy of the fluid. Then, energy conservation reads [cf. (*)]

$$\frac{d}{dt}\int_V dV\left(\frac{1}{2}\rho u^2 + e\right) = \int_V dV \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e\right) = \underbrace{-\int_{\partial V} d\underline{S}\cdot\left(\frac{1}{2}\rho u^2 + e\right)\underline{u}}_{\text{fluid flow carrying energy through boundary}}$$

$$\underbrace{-\int_{\partial V} d\underline{S}\cdot p\underline{u} + \int_{\partial V} d\underline{S}\cdot\underline{\underline{\sigma}}\cdot\underline{u} + \int_V dV\, \underline{f}\cdot\underline{u}}_{\text{work done by forces}} \underbrace{- \int_{\partial V} d\underline{S}\cdot\underline{q}}_{\text{heat flux}}$$

Once again using the divergence theorem and the arbitrariness of V, we deduce that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e\right) = -\underline{\nabla}\cdot\left[\underline{u}\left(\frac{1}{2}\rho u^2 + p + e\right)\right] + \underline{\nabla}\cdot(\underline{\underline{\sigma}}\cdot\underline{u})$$
$$+ \underline{f}\cdot\underline{u} - \underline{\nabla}\cdot\underline{q}$$

If $\underline{f}$ is conservative, i.e. $\underline{f} = -\rho\underline{\nabla}\Phi$, then we can make a further simplification:

$$\underline{f}\cdot\underline{u} = -\rho\underline{u}\cdot\underline{\nabla}\Phi = -\underline{\nabla}\cdot(\rho\underline{u}\Phi) + \Phi\,\underbrace{\underline{\nabla}\cdot(\rho\underline{u})}_{\text{// continuity eqn.}}$$

$$-\Phi\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\rho\Phi) + \rho\frac{\partial\Phi}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi\right) + \underline{\nabla}\cdot\left[\rho\underline{u}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \frac{e}{\rho} + \Phi\right)\right] = \underline{\nabla}\cdot(\underline{\underline{\sigma}}\cdot\underline{u}) - \underline{\nabla}\cdot\underline{q} + \rho\frac{\partial\Phi}{\partial t}}$$

3. Constitutive relations

The three boxed equations are the fundamental equations of fluid mechanics. However, in general, we need to supply additional information in order to solve them. In particular, we need:

- $e(p,\rho)$, i.e., an expression for the internal-energy density. For an ideal gas,

$$e(p,\rho) = e(p) = \frac{p}{\gamma - 1}$$

where γ = ratio of specific heats, "adiabatic index".

<u>Exercise</u> show that for $e = p/(\gamma - 1)$, with $\underline{q} = \underline{\underline{\sigma}} = 0$,

$$\frac{d}{dt}\frac{p}{\rho^\gamma} = \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \underline{u}\cdot\underline{\nabla}\left(\frac{p}{\rho^\gamma}\right) = 0 .$$

This is the fluids equivalent of the ~~ideal~~ adiabatic relation $pV^\gamma = \text{const}$ for ideal gases.

- A prescription for $\underline{q}$ and $\underline{\underline{\sigma}}$.

$\underline{q}$: In a gas, $\underline{q} = -\kappa \underline{\nabla} T$, where T is the temperature, specified by an <u>equation of state</u>, which we also have to provide. For an ideal gas, $p = \frac{k_B}{m}\rho T$, and κ can also be calculated from kinetic theory, but in general both have to be measured experimentally.

$\underline{\underline{\sigma}}$: We can constrain the form of $\underline{\underline{\sigma}}$ with the following observations:

(i) Internal friction occurs only when different (neighbouring) fluid particles move with different velocities $\Rightarrow$ $\underline{\underline{\sigma}}$ depends on spatial derivatives of $\underline{u}$.

(ii) If these derivatives are small, we may suppose that $\underline{\underline{\sigma}}$ depends <u>linearly</u> on only the <u>first</u> derivatives, $\partial u_i/\partial x_j$.

iii) when the whole fluid is in uniform rotation, there is no internal friction, so $\underline{\underline{\sigma}}$ vanishes if $\underline{u} = \underline{\Omega} \times \underline{r}$.

The most general form that $\underline{\underline{\sigma}}$ can take subject to these constraints is

$$\sigma_{ij} = \underset{\uparrow}{\mu} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \underset{\uparrow}{\zeta} \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

dynamic viscosity or "first viscosity"

bulk viscosity or "second viscosity".

Neither viscosity depends on $\underline{u}$ (but either can depend on ρ or p). Their values must be measured experimentally or determined by kinetic theory — significantly, it may be shown from the latter that $\zeta = 0$ for an ideal monatomic gas.

Substituting this form for $\underline{\underline{\sigma}}$ into the momentum equation derived earlier, we find that

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla} \underline{u} \right) = - \underline{\nabla} p + \underline{\nabla} \cdot \left\{ \mu \left[\underline{\nabla} \underline{u} + (\underline{\nabla} \underline{u})^T - \frac{2}{3} \underline{\nabla} \cdot \underline{u} \, \underline{\underline{I}} \right] + \zeta \underline{\nabla} \cdot \underline{u} \, \underline{\underline{I}} \right\} + \underline{f}$$

This very important equation is known as the "Navier-Stokes equation".

In the rest of the lecture, we shall explore some of the (many) ~~qualitative~~ features of the equations derived in Section 2 that are independent of constitutive relations like our choice of $e(\rho, s)$ and the equation of state.

4. Bernoulli's equation

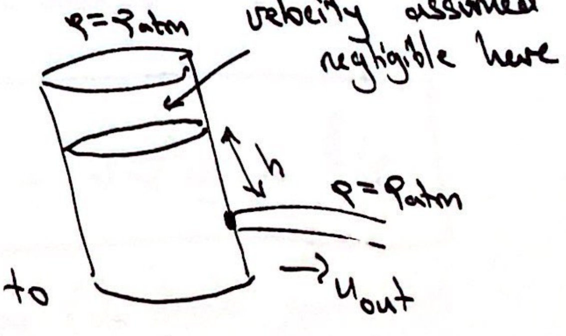
From the energy equation (boxed on p.6) with $\underline{q} = \underline{\underline{\sigma}} = 0$ and $\frac{\partial}{\partial t} = 0$ (steady-flow assumption), it follows that

$$\rho \underline{u} \cdot \underline{\nabla} \left[\frac{1}{2} u^2 + \frac{p}{\rho} + \frac{e}{\rho} + \Phi \right] = 0 .$$

$$\Rightarrow \quad \frac{1}{2} u^2 + \frac{p}{\rho} + \frac{e}{\rho} + \Phi = \text{constant along lines of flow}$$
("streamlines")

An example of its use is to derive "Toricelli's law", which relates the speed of fluid flowing out of a hole in a container to the height of the fluid below the hole. $\Phi = gz$, so Bernoulli gives (w/ const density and pressure)

$$\frac{1}{2} u_{out}^2 = gh \quad \Rightarrow \quad u_{out} = \sqrt{2gh} .$$



Bernoulli's equation can also be used to explain why flow past an aerofoil generates lift.

5. Incompressible fluids

As we have just seen, many fluid-mechanical problems are simplified if we can assume that the density of the fluid is constant. Such a fluid is called "incompressible".

A fluid flow can be called incompressible if

$$1 \gg \frac{\Delta \rho}{\rho} \sim \frac{\Delta p}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{u^2}{c_s^2}$$

$\sim \rho u^2$ by Bernoulli's eqn.

const. entropy, i.e. ideal conditions

where $c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$ is the sound speed — see tomorrow's lecture on waves. We can therefore consider a flow to be incompressible, i.e., $\rho \approx$ const, if the velocity u is ~~small~~ sub-sonic.

The equations of fluid mechanics undergo a remarkable simplification if ρ = const. The continuity equation $\frac{d}{dt}\rho = -\rho \underline{\nabla} \cdot \underline{u} = 0$ $\Rightarrow \boxed{\underline{\nabla} \cdot \underline{u} = 0}$, while the Navier-Stokes equation reduces to

$$\boxed{\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\, \underline{u} = -\underline{\nabla}\left(\frac{p}{\rho}\right) + \nu \nabla^2 \underline{u} + \underline{f}/\rho}$$

where $\nu = \mu/\rho$ is the "kinematic viscosity" (do check you can derive this!).

A further enormous simplification follows from the fact that the energy equation has become redundant: the boxed system on the previous page is closed, because pressure can be computed by solving the Poisson-type equation

$$-\nabla^2\left(\frac{p}{\rho}\right) = \underline{\nabla} \cdot \left[\underline{u} \cdot \underline{\nabla}\, \underline{u}\right].$$

Because the boxed system is closed, its only free parameter is the kinematic viscosity (provided f is already specified). ~~Thus~~ Thus, the response of an incompressible fluid to an external force essentially only depends on the relative size of the viscous term compared with the other terms in the incompressible Navier-Stokes equation, which is quantified by the Reynolds <u>number</u>

$$Re \equiv \frac{|\underline{u} \cdot \underline{\nabla}\, \underline{u}|}{|\nu \nabla^2 \underline{u}|} \sim \frac{UL}{\nu}.$$

Loosely, flows with $Re \ll 1$ are <u>laminar</u>

while flows with $Re \gg 1$ are <u>turbulent</u>

You will learn much more about this in a later lecture.