Note: There is much more here than you could possibly do in a week or a month or perhaps even a year, no matter your background. But you can take these problems with you and learn from them over time. To guide you, each problem is given a ski-slope rating according to its intended difficulty: ●, ■, ◆, ◆◆.

Hydrodynamics

1. ● **Shrinking sink streams.** Go to the bathroom and turn on the sink slowly to get a nice, laminar stream flowing down from the faucet. Go on, I'll wait. If you followed instructions, then you'll see that the stream becomes more narrow as it descends. Knowing that the density of water is very nearly constant, use the continuity equation to show that the cross-sectional area of the stream $A(z)$ as a function of distance from the faucet z is

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/v_0^2}},$$

where A_0 is the cross-sectional area of the stream upon exiting the faucet with velocity v_0 and g is the gravitational acceleration. If you turn the faucet to make the water flow faster, what happens to the tapering of the stream?

2. ● **Self-gravity is stressful.** Use Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, to show that the gravitational force on a self-gravitating fluid element may be written as

$$-\rho\boldsymbol{\nabla}\Phi = -\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{\boldsymbol{gg}}{4\pi G} - \frac{g^2}{8\pi G}\boldsymbol{I}\right),$$

where $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, $g^2 = \boldsymbol{g}\boldsymbol{\cdot}\boldsymbol{g}$, $\boldsymbol{I}$ is the unit dyadic, and G is Newton's gravitational constant. The quantity inside the divergence operator is known as the gravitational stress tensor. Written in the form of a divergence, the gravitational force represents the flux of total momentum through a surface due to gravitational forces.

3. ● **Straining in cylindricals.** Show that the $R\varphi$-component in cylindrical coordinates of the rate-of-strain tensor

$$W_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

is given by

$$W_{R\varphi} = \frac{1}{R}\frac{\partial u_R}{\partial \varphi} + R\frac{\partial}{\partial R}\frac{u_\varphi}{R}.$$

(Hint: $\partial u_i/\partial x_j = [(\hat{\boldsymbol{e}}_j\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u}]\boldsymbol{\cdot}\hat{\boldsymbol{e}}_i$ is coordinate invariant.) Such a combination often shows up in the theory of angular-momentum transport in accretion discs.

4. ■ **Helicity conservation.** Given the vorticity $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$, the helicity of a region of fluid is defined to be $\mathcal{H} \doteq \int \mathrm{d}\mathcal{V}\,\boldsymbol{\omega}\boldsymbol{\cdot}\boldsymbol{u}$, where the integral is taken over the volume of that region. Assume that the circulation $\Gamma = \text{const}$ and that $\boldsymbol{\omega}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ vanishes over the surface bounding $\mathcal{V}$,

where $\hat{\boldsymbol{n}}$ is the unit normal to that surface. Prove that the helicity $\mathcal{H}$ is conserved in a frame moving with the fluid, *viz.* $\mathrm{D}\mathcal{H}/\mathrm{D}t = 0$. Note that the fluid need not be incompressible for this property to hold.

5. ◆ **Spiral density waves and inertial waves.** Section VI.4 of Kunz's lecture notes contains a linear analysis of an unmagnetized, adiabatic, self-gravitating fluid. With P and ρ being the background thermal pressure and mass density (both taken to be uniform), the dispersion relation governing small-amplitude perturbations was $\omega^2 - k^2a^2 + 4\pi G\rho = 0$ with $a^2 \doteq \gamma P/\rho$. Solutions were "Jeans unstable" for $k^2a^2 < 4\pi G\rho$. This problem has you repeat this linear analysis, but in cylindrical coordinates (R, φ, z) for a differentially rotating disk with angular frequency $\boldsymbol{\Omega} = \Omega(R)\hat{\boldsymbol{z}}$. Your starting point will be §II.5 of Kunz's lecture notes, where you will find the hydrodynamic equations written in a rotating frame. In what follows, take the background pressure to be barotropic and allow the background $\rho = \rho(R, z)$.

(a) Take the perturbations to have space-time dependence $\exp(-\mathrm{i}\omega t + \mathrm{i}m\varphi + \mathrm{i}k_R R + \mathrm{i}k_z z)$ with $k_R L_R \sim k_R R \gg 1$ and $k_z L_z \gg 1$, where L_R (L_z) is the characteristic disk lengthscale in the radial (vertical) direction. (This is a WKB approximation: the perturbations are assumed to vary on lengthscales much shorter than those characterizing the background.) Obtain the following dispersion relation in the "tightly wound" limit in which both k_R and $k_z \gg m/R$:

$$\varpi^4 - \varpi^2\bigl(\kappa^2 + k^2a^2 - 4\pi G\rho\bigr) + \kappa^2\frac{k_z^2}{k^2}\bigl(k^2a^2 - 4\pi G\rho\bigr) = 0,$$

where $\varpi \doteq \omega - m\Omega$ is a Doppler-shifted frequency, $\kappa^2 = 4\Omega^2 + \mathrm{d}\Omega^2/\mathrm{d}\ln R$ is the square of the epicyclic frequency, and $k^2 = k_R^2 + k_z^2$. Another way to write this result is

$$\varpi^2 - k^2a^2 + 4\pi G\rho = \frac{\kappa^2 k_R^2 a^2}{\varpi^2 - \kappa^2}\left(1 - \frac{4\pi G\rho}{k^2a^2}\right),$$

which has the usual Jeans dispersion relation on the left-hand side (but for $\omega^2 \to \varpi^2$) and has a right-hand side that includes effects associated with the differential rotation.

(b) Consider the case $k_z = 0$. The result is the dispersion relation for *spiral density waves*:

$$\varpi^2 = \kappa^2 + k^2a^2 - 4\pi G\rho.$$

Such waves are thought to be particularly important in theories of galactic structure and protostellar disks. Note that rotation is a stabilizing influence (as is differential rotation if $\kappa^2 > 0$, the usual situation in astrophysical disks). Physically, why?

(c) Now take $k_z a \gg \kappa$ and $(4\pi G\rho)^{1/2}$ to obtain the dispersion relation for *inertial waves*:

$$\varpi^2 = \frac{k_z^2}{k^2}\kappa^2.$$

These waves are essentially incompressible, and are the only fluctuations in a polytropic, non-self-gravitating disk with frequencies less than κ. Note the dependence on k_z, which in concert with their incompressible nature tells us that the fluid displacements in this wave are primarily in the disk plane. With that in mind, what force is responsible for this wave?

(d) ◆◆ Repeat the calculation in part (a) but *without* adopting the WKB approximation. Namely, take the perturbations to have the form $f(R,z)\exp(-\mathrm{i}\omega t + \mathrm{i}m\varphi)$ and obtain the following linear wave equation for the potential $\delta h \doteq \delta P/\rho + \delta\Phi$:

$$\left[\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\rho}{D}\frac{\partial}{\partial R}\right) - \frac{1}{\varpi^2}\frac{\partial}{\partial z}\left(\rho\frac{\partial}{\partial z}\right) - \frac{m^2\rho}{R^2 D} + \frac{1}{\varpi R}\frac{\partial}{\partial R}\left(\frac{2\Omega m\rho}{D}\right)\right]\delta h = -\delta\rho,$$

where $D \doteq \kappa^2 - \varpi^2$ and $\delta\Phi$ is the solution to the linearized Poisson's equation. Note the resonances at $D = 0$ and $\varpi = 0$, which are referred to as the Lindblad and corotation resonances, respectively. Near these resonances, the waves couple strongly to the disk. (The WKB treatment formally breaks down at the Lindblad resonance, at which k_R must vanish.) These resonances are important in the study of tidally driven waves and planetary migration. For more on this topic, see Goldreich & Tremaine (1979, *Astrophys. J.* **233**, 857) and Balbus (2003, *Annu. Rev. Astron. Astrophys.* **41**, 555).

Magnetohydrodynamics: Waves

6. ● **A mechanical Alfvén wave.** Suppose we have a perfectly conducting rectangular loop of height h and part of its width x immersed in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ oriented out of the page. The loop has a mass m and inductance L. Ignore gravity.

(a) Give the loop an initial velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{x}}$ to the right, so that the flux through the loop increases in time. What happens? Describe the motion in words.

(b) Solve for the motion analytically.

(c) Now suppose that the loop has some resistance R. How big should R be before resistance plays an appreciable role in the motion?

7. ● **Transport of energy by an Alfvén wave.** A circularly polarized Alfvén wave of amplitude $\delta B_\perp$ propagates along an otherwise uniform magnetic field $B_0\hat{\boldsymbol{z}}$:

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta B_\perp\,\boldsymbol{e}_\perp(t,z) \quad \text{and} \quad \boldsymbol{u} = -\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\,\boldsymbol{e}_\perp(t,z), \tag{1}$$

where

$$\boldsymbol{e}_\perp(t,z) = \cos[k(v_\mathrm{A}t - z)]\hat{\boldsymbol{x}} + \sin[k(v_\mathrm{A}t - z)]\hat{\boldsymbol{y}}.$$

(a) Draw the magnetic-field line at $t = 0$. Which way is the wave propagating?

(b) Prove that the magnetic-field strength B is a constant, despite the presence of the wave.

(c) Show that (1) is an exact nonlinear solution of the ideal-MHD equations.

(d) Calculate the time-averaged Poynting flux $\langle\boldsymbol{S}\rangle_t \doteq \langle c\boldsymbol{E}\times\boldsymbol{B}/4\pi\rangle_t$ for this wave. Write it in terms of the total wave energy $\mathcal{E} = \rho u^2/2 + \delta B_\perp^2/8\pi$. Interpret your result physically.

Magnetohydrodynamics: Conservation laws

8. ● **Kelvin's circulation theorem in MHD**. In §II.4 of Kunz's lecture notes, Kelvin's circulation theorem was proven for the case without a magnetic field. Here you will generalize it for MHD. First, a reminder of the hydrodynamic case:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} \doteq \frac{\mathrm{D}}{\mathrm{D}t}\int_{\partial\mathcal{S}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell} = \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\mathcal{S}} = \oint_{\partial\mathcal{S}} \left(-\frac{1}{\rho}\boldsymbol{\nabla}P\right)\cdot\mathrm{d}\boldsymbol{\ell} = -\oint_{\partial\mathcal{S}} \frac{\mathrm{d}P}{\rho},$$

where $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity. For a barotropic fluid, $\Gamma = \mathrm{const}$ in the frame of the flow.

(a) Use the MHD force equation to show that Kelvin's circulation theorem in MHD becomes

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_{\partial\mathcal{S}} \left(-\frac{\mathrm{d}P}{\rho} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)\cdot\mathrm{d}\boldsymbol{\ell},$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.

(b) Explain how the Lorentz force could generate circulation. (Hint: Take an irrotational fluid and thread it with a twisted magnetic field. Let it go. What would happen?) Would it help or hurt vorticity conservation if the magnetic field weren't perfectly frozen into the plasma? Why?

9. ■ **Lundquist's theorem.** The concept of flux freezing is usually introduced by way of Alfvén's theorem: the magnetic flux passing through a surface moving along with the fluid is conserved. There is an alternative description of flux freezing stated in terms of line tying: fluid elements that lie on a field line initially will remain on that field line (S. Lundquist, *Phys. Rev.* **83**, 2 (1951)). Starting from the ideal induction equation, $\partial\boldsymbol{B}/\partial t = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})$, use the continuity equation to show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\boldsymbol{B}}{\rho} = \frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\boldsymbol{u},$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. By comparing this equation to that describing the evolution of an infinitesimal Lagrangian separation vector between two points in a moving fluid, argue that the magnetic field moves with the flow.

Magnetohydrodynamics: Instabilities

10. ◆ **Magnetorotational instability with springs**. The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led Balbus and Hawley to develop the now-famous spring model of the MRI, which was then used to conjecture that the Oort A-value is the universal growth rate limit for accretion-disk shear instabilities. The Hill equations describe local disk dynamics in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\varphi - \Omega_0 t)$ from a circular orbit $R = R_0$, $\varphi = \Omega_0 t$. They are given by:

$$\ddot{x} - 2\Omega_0\dot{y} = -4A_0\Omega_0 x + f_x, \tag{2a}$$

$$\ddot{y} + 2\Omega_0\dot{x} = f_y, \tag{2b}$$

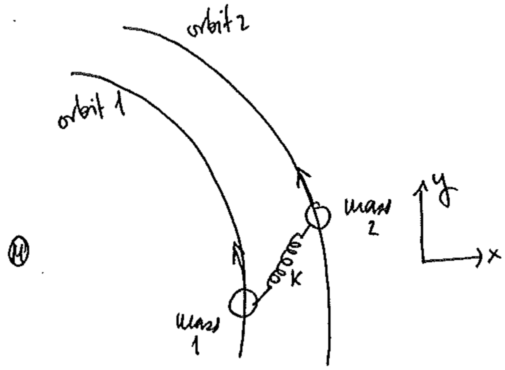
where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The Oort A-value $A_0 = -(3/4)\Omega_0$ for Keplerian rotation.[1]

The MRI analogy goes as follows. Consider the local force to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear; for an *isotropic* force, we have $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is some constant. (You could also profitably think of this force as being due to an ideal spring with spring constant K.) Then (2) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x - Kx, \tag{3a}$$

$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{3b}$$

Visually,



Now then...

(a) For small displacements x, y, show that the solutions to (3) are $\propto \exp(\pm \mathrm{i}\omega t)$ with

$$\omega^4 - \omega^2\big(\kappa^2 + 2K\big) + K\big(K + 4A_0\Omega_0\big) = 0, \tag{4}$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency, which is positive for Keplerian rotation. Equation (4) should look familiar from the lecture notes on MHD instabilities: set $K = 0$ and you get trivial displacements ($\omega^2 = 0$) and epicycles ($\omega^2 = \kappa^2$); replace K with $(\boldsymbol{k} \cdot \boldsymbol{v}_\mathrm{A})^2$ and you get the axisymmetric MRI linear dispersion relation. Show that $A_0 < 0$ is a necessary (but not sufficient) condition for instability.

(b) S. A. Balbus and J. F. Hawley, *Astrophys. J.* **392**, 662 (1992) conjecture "that the Oort A-value is an upper bound to the growth rate of any instability feeding upon the free energy of differential rotation." En route, they show that the maximum growth rate of the MRI is the Oort-A value, that it occurs at $K_\mathrm{max}/\Omega_0^2 = -(A_0/\Omega_0)(2 + A_0/\Omega_0)$, and that the corresponding eigenvector satisfies $y/x = -1$, i.e., radial and azimuthal displacements are equal in size. Prove these three facts.

(c) Use these to show that, at maximum growth, the Lagrangian change in the rotation frequency of a displaced fluid element is $\Delta\Omega = \dot{y}/R_0 = -|A_0|x/R_0$ and that the corresponding Lagrangian change in its specific angular momentum $\ell = \Omega R^2$ satisfies

$$\frac{\Delta\ell}{\ell_0} = 2\frac{x}{R_0} + \frac{\Delta\Omega}{\Omega_0} = 2\left(1 - \frac{|A_0|}{2\Omega_0}\right)\frac{x}{R_0}.$$

[1]The notation for differential rotation varies in the accretion-disk literature; here's a dictionary: $2A_0 = -q\Omega_0 = (\mathrm{d}\Omega/\mathrm{d}\ln R)_{R=R_0}$. Often, the "0" subscript is simply dropped for ease of notation.

Then show that outwardly (inwardly) displaced fluid elements always have more (less) angular momentum that the orbits they are passing through (which is what makes instability possible). (Hint: what is the difference in ℓ between two undisturbed orbits a radial distance x apart, in a disk in which $\mathrm{d}\Omega/\mathrm{d}\ln R = 2A_0 < 0$?)

(d) **Bonus**. Set $f_x = -K_x x$ and $f_y = -K_y y$ with $K_x \neq K_y$ being positive constants. Compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate $\gamma_{\max}$ and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\max}$ is achieved. (It may help to make a quick contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.) E. Quataert, W. Dorland, and G. W. Hammett *Astrophys. J.* **577**, 524 (2002) used this as a model for the magnetorotational instability in a collisionless, magnetized plasma.

Magnetohydrodynamics: Relaxation

11. ◆ **Wöltjer–Taylor relaxation**. In some systems (e.g., the solar corona, experiments in plasma confinement using a toroidal pinch), the plasma evolves towards a preferred configuration known as the "relaxed state". This state is in a configuration of minimum magnetic energy, but a minimum energy subject to the constraint that the global magnetic helicity $H_0 \doteq \int_{\mathcal{V}_0} \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B}$ is conserved. (Here, $\boldsymbol{A}$ is the vector potential satisfying $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and $\mathcal{V}_0$ is the total volume of the isolated plasma under consideration). Helicity can be interpreted in a topological sense as the number of linkages of magnetic flux tubes with one another; you can read about this in pretty much any decent textbook on MHD. Even when the plasma is not ideal, helicity conservation seems to remain a fairly good approximation.[2]

(a) Show that, while $\boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B}$ is not gauge invariant, its integral within a flux tube is. (Hint: recall from undergraduate electromagnetism that $\boldsymbol{A} \to \boldsymbol{A}+\boldsymbol{\nabla}\psi$, where ψ is an arbitrary scalar function, changes nothing in Maxwell's equations.) State under what conditions H_0 is gauge invariant.

(b) Show that H_0 is a conserved quantity in ideal MHD (but not in resistive MHD).

(c) ◆◆ Use the variational principle to minimize magnetic energy subject to constant helicity:

$$\delta \int \mathrm{d}^3\boldsymbol{r}\, \big(B^2 - \alpha \boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B}\big) = 0,$$

where α is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Show that this procedure yields $\boldsymbol{\nabla}\times\boldsymbol{B} = \alpha\boldsymbol{B}$ (and thus $\nabla^2\boldsymbol{B} = -\alpha^2\boldsymbol{B}$, the Helmholtz equation), i.e., $\boldsymbol{B}$ is a linear force-free field. What boundary conditions must

[2] Some history: Wöltjer (1958) showed that there are an infinite number of integral invariants in ideal MHD: $H_i \doteq \int_{\mathcal{V}_i} \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B} = \mathrm{const}$ on each and every flux tube $\mathcal{V}_i$ in the system. These invariants are related to the well-known property that the magnetic field is frozen into an ideally conducting plasma. J. B. Taylor (1974) realized that, in a slightly resistive turbulent plasma contained within a perfectly conducting boundary, the only flux tube to retain its integrity is that which contains the entire plasma. Then, only H_0 will remain invariant. Taylor's conjecture is that MHD systems tend to minimize their magnetic energy subject to the constraint that the *total* magnetic helicity remains constant.

you impose to obtain this result? You may find it helpful to use flux freezing in the Lagrangian viewpoint, *viz.*, $\delta \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})$, where $\boldsymbol{\xi}$ is the Lagrangian displacement of a fluid element (see §VI.3 of Kunz's lecture notes).

(d) Consider a relaxed (i.e., linear force-free) field with cylindrical symmetry: $\partial/\partial\varphi = 0$, $\partial/\partial z = 0$. Show that $B_z = B_0 J_0(\alpha R)$ and $B_\varphi = B_0 J_1(\alpha R)$, where J_n is the nth Bessel function and R is the cylindrical radius. This corresponds to a field twisted about a cylindrical surface ("cylindrical pinch").

If you're interested in learning more, consult J. B. Taylor (1986), RvMP, 58, 741.

MHD equilibria

12. **Magnetostatic stars.** A self-gravitating, non-rotating star contains an axisymmetric and purely toroidal magnetic field $\boldsymbol{B} = B(R, z)\hat{\boldsymbol{\varphi}}$, where (R, φ, z) are cylindrical polar coordinates. For this configuration, show that the equation of magnetohydrostatic equilibrium can be written as

$$0 = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}P - \frac{B}{4\pi\rho R}\boldsymbol{\nabla}(RB), \tag{5}$$

where Φ is the self-gravitational potential, ρ is the mass density, and P is the gas pressure. Then take the gas pressure to be barotropic, $P = P(\rho)$, and use equation (5) to show that the magnetic field must then satisfy

$$B = \frac{1}{R}F(\rho R^2),$$

where F is an arbitrary function. (Fun fact: Cowling's anti-dynamo theorem says that an axisymmetric magnetic field that vanishes at infinity cannot be maintained by dynamo action.)

MHD shocks

13. **Earth's bow shock.** (Based on a problem from Thorne & Blandford) The Sun leaks $\sim 10^{-14}$ $\mathrm{M}_\odot$ yr^{-1} off its surface in the form of a supersonic, hydromagnetic flow of plasma. At the radius of the Earth's orbit (1 au), this "solar wind" is characterized by a bulk velocity $v \sim 400$ km s^{-1}, density $n \sim 10$ cm^{-3}, and temperature $T \sim 10^5$ K. It is threaded by an interplanetary magnetic field, arranged approximately in the shape of a spiral emerging from the Sun (as predicted by E. Parker), whose strength at 1 au is $B \sim 50$ μG.

(a) Balance the momentum flux of the solar wind with the magnetic pressure exerted by the Earth's dipolar magnetic field to estimate the radius above the Earth at which the solar wind passes through a bow shock. (Useful facts: The strength of the Earth's magnetic field at the surface is $B_\oplus \sim 0.5$ G. The radius of the Earth is $R_\oplus = 6371$ km.)

(b) Consider a strong perpendicular shock at which the magnetic field is parallel to the shock front. Show that the magnetic-field strength will increase by the same ratio as the density when crossing the shock front. Is there an upper limit to the factor by which a perpendicular shock can increase the magnetic field?

Turbulence

14. ◆ **Critical balance.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. (These are the "inertial waves" seen in Problem 5.) Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? Is the similar to or different than Goldreich–Sridhar turbulence?

Magnetic reconnection

15. ■ **2D magnetic reconnection?** Consider a plasma that is rigorously described by the following resistive-MHD Ohm's law: $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B} = \eta\boldsymbol{j}$. Suppose that the velocity and magnetic fields are two-dimensional, with $\boldsymbol{u} = u_x(t,x,y)\hat{\boldsymbol{x}} + u_y(t,x,y)\hat{\boldsymbol{y}}$ and $\boldsymbol{B} = B_x(t,x,y)\hat{\boldsymbol{x}} + B_y(t,x,y)\hat{\boldsymbol{y}}$, respectively. Use Ampère's law to show that $\boldsymbol{j} = j_z(t,x,y)\hat{\boldsymbol{z}}$. Then show that there is a velocity $\boldsymbol{v}$ such that $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = 0$. Give an explicit expression for $\boldsymbol{v}$ in terms of $\boldsymbol{u}$, $\boldsymbol{j}$, and $\boldsymbol{B}$. Does this mean there is no reconnection in two dimensions? If not, why not?

15. ◆◆ **Simple solutions for shrinking sheets (and their not-so-simple tearing).** Under certain restrictions, one can use the MHD equations to obtain relatively simple, *time-dependent* solutions for a thinning current sheet (CS) and examine their linear stability to resistive tearing modes. These time-dependent solutions are based on S. Chapman & P. C. Kendall, *Proc. Roy. Soc. London Ser. A*, **271**, 435 (1963), and were used recently by N. F. Loureiro and D. A. Uzdensky, *Phys. Rev. Lett.* **116**, 105003 (2016) and E. A. Tolman *et al.*, *J. Plasma Phys.* **84**, 905840115 (2018) in their studies of the onset of reconnection in a thinning CS (the same E. A. Tolman who lectured at our summer school). Here's the basic idea...

(a) First, use that the incompressible MHD equations in two dimensions (x, y) to show that

$$\frac{\partial\Psi}{\partial t} + \{\Phi,\Psi\} = 0 \quad\text{and}\quad \frac{\partial}{\partial t}\nabla^2\Phi + \{\Phi,\nabla^2\Phi\} = \{\Psi,\nabla^2\Psi\}, \tag{6}$$

where the flux function Ψ defines the magnetic field $\boldsymbol{B} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\Psi$, the stream function Φ defines the flow velocity $\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\Phi$, and the Poisson bracket

$$\{\Phi,\Psi\} \doteq \frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial x}.$$

.

(b) Consider the following time-dependent stream and flux functions:

$$\Phi(t,x,y)=\Lambda(t)xy \quad \text{and} \quad \Psi(t,x,y)=\frac{B_0}{2}\left[\frac{x^2}{a(t)}-\frac{y^2}{L(t)}\right]. \tag{7}$$

These describe a local incompressible flow that is thinning and lengthening a reversing magnetic field about an X-point. For these potentials to be solutions of the 2D MHD equations, what must the CS width $a(t)$ and length $L(t)$ satisfy? Plot iso-contours of $\Phi/(\Lambda a^2)$ and $\Psi/(B_0 a)$ in the (x/a)–(y/a) plane for $L/a=10$ and describe what you see. (You might find it helpful to calculate $\boldsymbol{u}$ and $\boldsymbol{B}$ corresponding to these functions and plot their vector fields.)

(c) Set $\Lambda(t)=\tau^{-1}$ with $\tau=\text{const}$ and solve your equations for $a(t)$ and $L(t)$. (Name the initial values of the CS thickness and length a_0 and L_0, respectively.) Briefly describe in words the evolution of this CS.

(d) Suppose $L(t)=L_0(1+t/\tau)$ with $\tau=\text{const}$. Obtain the corresponding $\Lambda(t)$ and $a(t)$. Briefly describe in words the evolution of this CS.

(e) Let's adopt the CS model from part (d) and set $\tau\doteq(L_0/v_{\rm A})M_{\rm A}^{-1}$, where $M_{\rm A}$ is the Alfvén Mach number of the incompressible flow. We now ask how linear tearing modes grow on top of this time-dependent background and determine which of these linear modes grows the fastest at any given time in the CS evolution. For that, give the CS some resistivity η, and assume that the outer solution for the CS provides $\Delta'(k)\sim 1/ka^2$. (The "$\sim$" here means that we are dropping factors of order unity.) The number of tearing-induced magnetic islands with wavenumber k that can fit inside the length of this CS at any given time is $\sim kL\doteq N$. Because each tearing-mode wavelength k^{-1} is stretched by the flow in the same way as is L, each tearing mode can be labeled by its own unique value of N. With this borne in mind, answer the following:

(i) Take the long-time limit $t\gg\tau$, such that $L(t)\sim L_0(t/\tau)$. Write down how $\Delta'(N)$ evolves in time for this CS. Your answer should involve N, L_0, a_0, and t/τ only.

(ii) Show that, in the FKR regime, the time-dependent growth rate $\gamma_{\rm FKR}$ satisfies

$$\gamma_{\rm FKR}(t)\tau_0\sim N^{-2/5}M_{\rm A}^{12/5}S_0^{-3/5}\left(\frac{t}{\tau_0}\right)^{12/5}, \tag{8}$$

where $\tau_0\doteq(a_0L_0)^{1/2}/v_{\rm A}$ and $S_0\doteq v_{\rm A}(a_0L_0)^{1/2}/\eta$. Thus, the fastest-growing FKR mode is the $N=1$ mode.[3]

(iii) Use (8) to determine the approximate time at which the $N=1$ FKR mode grows faster than the rate at which the CS thickness is shrinking. (Don't be too fancy here – a scaling argument is enough.) Name this time $t_{\rm cr}$ and express it in terms of τ_0, $M_{\rm A}$, and S_0.

[3]In writing (8), we are implicitly assuming that the secular evolution of the CS does not greatly affect the instantaneous exponential growth of the tearing modes – only that it changes the instantaneous values of $\tau_{\rm A}$, τ_η, and $\Delta' a$ that figure into the usual FKR growth rate. This is a good approximation when $\gamma_{\rm FKR}(t)\gg|\dot{a}/a|$ – see Tolman *et al.* (2018) if you're interested in the more rigorous details.

(iv) Determine the approximate time at which this $N = 1$ mode transitions into the Coppi regime. Name this time $t_{\rm tr}$ and express it in terms of τ_0, $M_{\rm A}$, and S_0.

(v) For what combination of $M_{\rm A}$ and S_0 is $t_{\rm tr} \sim t_{\rm cr}$? In this situation, the maximally growing FKR mode enters the Coppi regime just as it begins growing fast enough to disrupt the evolving CS. Loureiro & Uzdensky argued that, under these conditions, this time marks the onset of reconnection and the disruption of the CS.[4]

(vi) Consider a solar flare powered by a reconnecting CS whose $L_0 \sim a_0 \sim 10^4$ km and which evolves according to our crude model here. Typical photospheric values are $v_{\rm A} \sim 2000$ km s^{-1}, $M_{\rm A} \sim 10^{-3}$, and $S_0 \sim 10^{13}$. If you plug these numbers in to your answer from part (v), you should find that $t_{\rm tr} \sim t_{\rm cr}$. Use this to estimate the time at which reconnection onsets, as well as the aspect ratio of the CS at this time. The former turns out to be reasonably consistent with the observed pre-flare energy-buildup times in the solar photosphere. Neat.

Charged particle motion

16. ● **Drift currents.**

(a) Write down expressions for the E-cross-B drift $\boldsymbol{v}_E$, the grad-B drift $\boldsymbol{v}_{\nabla B}$, the curvature drift $\boldsymbol{v}_{\rm c}$, and the polarization drift $\boldsymbol{v}_{\rm pol}$.

(b) In an ion–electron plasma, which of these drifts have currents associated with them?

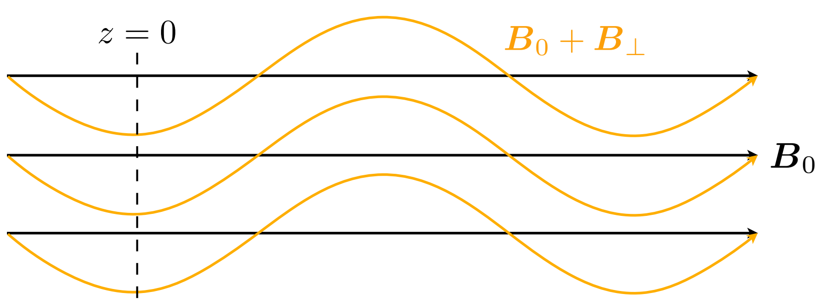
(c) Show that the current densities associated with the grad-B and curvature drifts are equal for a pressure-isotropic plasma in a force-free magnetic field having $\boldsymbol{j}\times\boldsymbol{B} = 0$. [Answer: $\boldsymbol{j}_{\nabla B} = \boldsymbol{j}_{\rm c} = (cP/B)\,\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B$.]

17. ■ **Drifts in MHD waves.**

(a) A small-amplitude linearly polarized Alfvén wave of amplitude $B_\perp$ and wavenumber $k > 0$ propagates along a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\perp \sin\bigl[k(z - v_{\rm A}t)\bigr]\hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{u} = -v_{\rm A}\frac{B_\perp}{B_0}\sin\bigl[k(z - v_{\rm A}t)\bigr]\hat{\boldsymbol{x}},$$

respectively, where $v_{\rm A} \doteq B_0/\sqrt{4\pi\rho}$ is the Alfvén speed. Neglecting terms of order $B_\perp^2$ and higher, compute all guiding-center drifts for this wave. Draw them for an electron on the figure below:

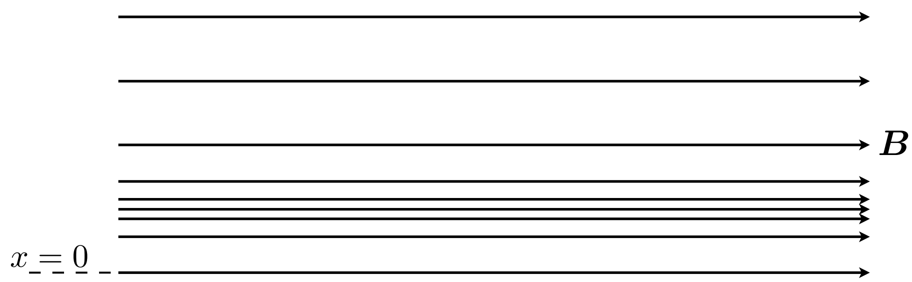


[4] They also considered the cases $t_{\rm tr} > t_{\rm cr}$ and $t_{\rm cr} < t_{\rm tr}$; I picked $t_{\rm tr} \sim t_{\rm cr}$ just to keep this problem short(ish).

(b) A small-amplitude fast mode of amplitude $B_\parallel$ and wavenumber $k > 0$ propagates across a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\parallel \sin\big[k(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{z}} \quad \text{and} \quad \boldsymbol{u} = v_\mathrm{f}\frac{B_\parallel}{B_0}\sin\big[k(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{x}},$$

respectively, where $v_\mathrm{f} \doteq \sqrt{v_\mathrm{A}^2 + a^2}$ is the fast magnetosonic speed. Neglecting terms of order $B_\parallel^2$ and higher, compute all guiding-center drifts for this wave. Draw them for an electron on the figure below:



18. **Drifts in dipoles.** The equation for a dipole magnetic field in spherical coordinates is given by

$$\boldsymbol{B} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} = \frac{m}{r^3}\Big(2\cos\vartheta\,\hat{\boldsymbol{r}} + \sin\vartheta\,\hat{\boldsymbol{\vartheta}}\Big)\,, \tag{9}$$

where $\boldsymbol{m} = m\hat{\boldsymbol{z}}$ is the magnetic moment.

(a) Show that the equation for a magnetic-field line is $r = R\sin^2\vartheta$, where R is the radius of the magnetic-field line at the equator ($\vartheta = \pi/2$).

(b) Show that the curvature of the magnetic-field line at the equator ($\vartheta = \pi/2$) is $R_\mathrm{c} = R/3$.

(c) Compute the curvature drift of a particle with charge q and parallel kinetic energy $W_\parallel$ at a radial distance R at the equator.

(d) Compute the grad-B drift of a particle with charge q and perpendicular kinetic energy $W_\perp$ at a radial distance R at the equator. For what ratio $W_\perp/W_\parallel$ are the drifts the same?

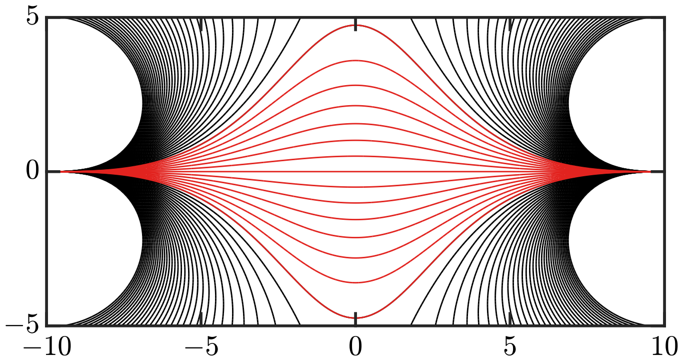
Now suppose there are two aligned magnetic dipoles with moment $\boldsymbol{m}$ spatially separated by $2\boldsymbol{a}$ about the origin. Using (9), the resulting magnetic field is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \left[\frac{3\boldsymbol{r}_+(\boldsymbol{m}\cdot\boldsymbol{r}_+)}{r_+^5} - \frac{\boldsymbol{m}}{r_+^3}\right] + \left[\frac{3\boldsymbol{r}_-(\boldsymbol{m}\cdot\boldsymbol{r}_-)}{r_-^5} - \frac{\boldsymbol{m}}{r_-^3}\right], \tag{10}$$

where $\boldsymbol{r}_\pm \doteq \boldsymbol{r} \pm \boldsymbol{a}$. This field may be obtained by taking the curl of the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{m}\times\boldsymbol{r}_+}{r_+^3} + \frac{\boldsymbol{m}\times\boldsymbol{r}_-}{r_-^3}. \tag{11}$$

Because $\partial\boldsymbol{A}/\partial t = \boldsymbol{0}$, we have $\boldsymbol{E} = \boldsymbol{0}$. Some magnetic-field lines in the $z = 0$ plane, obtained from the isocontours of A_y, are shown below, with those in red revealing a magnetic bottle:

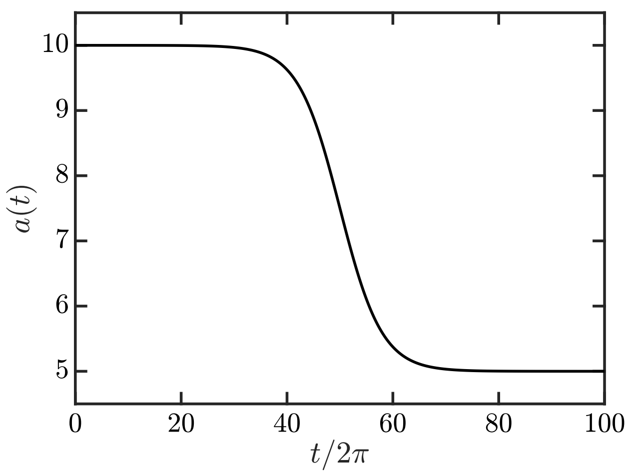


(e) Place a particle in the center of the mirror and launch it with velocity $\boldsymbol{v}$. Discuss how the particle moves for various initial pitch angles, $v_x(0)/v(0)$.

(f) Suppose the distance between the two dipoles in part (e) is adiabatically shrunk in half:

$$\boldsymbol{a} \to \boldsymbol{a}(t) = 10\hat{\boldsymbol{x}} - 2.5\big\{1 + \tanh[\gamma(t - t_{\mathrm{f}}/2)]\big\}\hat{\boldsymbol{x}},$$

with $\gamma \ll 1$, as shown in the figure below:



The vector potential defined by equation (11) then depends upon time, $\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(t, \boldsymbol{r})$, and so there is a non-zero electric field, $\boldsymbol{E}(t, \boldsymbol{r}) = -\partial \boldsymbol{A}/\partial t$. Discuss how the particle will move if $\boldsymbol{v}(0) = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})/\sqrt{2}$ (i.e., an initial pitch angle of 45°). In particular, what will $v_\parallel = \boldsymbol{v} \cdot \hat{\boldsymbol{b}}$ look like versus time?

Kinetics

19. ◆◆ **Weibel instability.** Consider a collisionless, unmagnetized, uniform plasma whose velocity distribution function of each species α is anisotropic with respect to the $\hat{\boldsymbol{z}}$ direction of a Cartesian coordinate system:

$$f_{0\alpha}(v_\parallel, v_\perp) = \frac{n_\alpha}{\sqrt{\pi} v_{\mathrm{th}\parallel\alpha}} \exp\left(-\frac{v_\parallel^2}{v_{\mathrm{th}\parallel\alpha}^2}\right) \frac{1}{\pi v_{\mathrm{th}\perp\alpha}^2} \exp\left(-\frac{v_\perp^2}{v_{\mathrm{th}\perp\alpha}^2}\right), \tag{12}$$

where $v_\parallel = \boldsymbol{v}\cdot\hat{\boldsymbol{z}}$, $v_\perp = |\boldsymbol{v}_\perp|$, and $\boldsymbol{v}_\perp = \boldsymbol{v} - v_\parallel\hat{\boldsymbol{z}}$. This *bi-Maxwellian* distribution function is characterized by the number density n_α and two different temperatures: the *parallel temperature* $T_{\parallel\alpha} \doteq m_\alpha v^2_{\mathrm{th}\parallel\alpha}/2$ and the *perpendicular temperature* $T_{\perp\alpha} \doteq m_\alpha v^2_{\mathrm{th}\perp\alpha}/2$. Into this plasma, introduce small-amplitude electromagnetic fluctuations as follows:

$$\begin{aligned}
f_\alpha(t,z,\boldsymbol{v}) &= f_{0\alpha}(v_\parallel,v_\perp) + \delta f_\alpha(\boldsymbol{v})\exp(-\mathrm{i}\omega t + \mathrm{i}kz),\\
\boldsymbol{E}(t,z) &= 0 + \delta\boldsymbol{E}\ \exp(-\mathrm{i}\omega t + \mathrm{i}kz),\\
\boldsymbol{B}(t,z) &= 0 + \delta\boldsymbol{B}\ \exp(-\mathrm{i}\omega t + \mathrm{i}kz),
\end{aligned}$$

where ω is the (complex) frequency and $k>0$ is the wavenumber. With page 30 of the NRL formulary in hand, answer the following:

(a) Retaining only terms linear in the fluctuation amplitudes, use the Vlasov and Maxwell equations to show that

$$\delta f_\alpha = \frac{\mathrm{i}}{k}\left[\frac{\hat{\boldsymbol{z}}}{v_\parallel - \omega/k}\frac{\partial f_{0\alpha}}{\partial v_\parallel} + \frac{k\boldsymbol{v}_\perp}{\omega}\left(\frac{1}{v_\parallel-\omega/k}\frac{\partial f_{0\alpha}}{\partial v_\parallel} - \frac{1}{v_\perp}\frac{\partial f_{0\alpha}}{\partial v_\perp}\right)\right]\cdot\frac{q_\alpha}{m_\alpha}\delta\boldsymbol{E}. \tag{13}$$

(b) For an *isotropic* Maxwellian $f_{0\alpha} = f_{0\alpha}(v)$ with $T_{\parallel\alpha} = T_{\perp\alpha} = T_\alpha$, equation (13) becomes

$$\delta f_\alpha = \frac{\mathrm{i}}{kv_\parallel - \omega}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\cdot\frac{q_\alpha}{m_\alpha}\delta\boldsymbol{E}.$$

Interpret this equation physically, paying particular attention to the pole at $\omega = kv_\parallel$.

(c) Use the appropriate Maxwell equations to show that

$$\delta\boldsymbol{E} = -\frac{4\pi\mathrm{i}}{\omega}\sum_\alpha q_\alpha\int\mathrm{d}^3\boldsymbol{v}\left[v_\parallel\hat{\boldsymbol{z}} + \boldsymbol{v}_\perp\left(1-\frac{k^2c^2}{\omega^2}\right)^{-1}\right]\delta f_\alpha. \tag{14}$$

(d) Combine equations (12)–(14) to derive the dispersion relation $\epsilon_\parallel(\omega,k)\epsilon_\perp(\omega,k) = 0$, where

$$\epsilon_\parallel(\omega,k) \doteq 1 + 2\sum_\alpha\frac{\omega^2_{\mathrm{p}\alpha}}{\omega^2}\zeta^2_\alpha\big(1+\zeta_\alpha Z(\zeta_\alpha)\big), \tag{15a}$$

$$\epsilon_\perp(\omega,k) \doteq 1 + \sum_\alpha\frac{\omega^2_{\mathrm{p}\alpha}}{\omega^2}\left(1-\frac{k^2c^2}{\omega^2}\right)^{-1}\left[\frac{T_{\perp\alpha}}{T_{\parallel\alpha}}\big(1+\zeta_\alpha Z(\zeta_\alpha)\big) - 1\right]. \tag{15b}$$

Here, $\omega^2_{\mathrm{p}\alpha} \doteq 4\pi q^2_\alpha n_\alpha/m_\alpha$ is the square of the plasma frequency, and $\zeta_\alpha \doteq \omega/|k_\parallel|v_{\mathrm{th}\parallel\alpha}$ is the dimensionless phase velocity that features as the argument of the plasma dispersion function $Z(\zeta_\alpha)$. (Some advice: Working in cylindrical coordinates with $\mathrm{d}^3\boldsymbol{v} = \pi\mathrm{d}v_\parallel\mathrm{d}v^2_\perp$ will simplify your velocity-space integrals considerably.)

(e) The solution $\epsilon_\parallel(\omega,k) = 0$ represents purely electrostatic fluctuations in the plasma, including the plasma oscillations and ion-acoustic waves. In a collisionless plasma, these fluctuations are Landau damped.

Our concern here is the solution $\epsilon_\perp(\omega,k) = 0$ for *electromagnetic* fluctuations. Take the plasma to be non-relativistic and to be composed of ions and electrons with $m_e/m_i \ll 1$.

Show that, in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

$$\gamma \approx \frac{k v_{\mathrm{th}\|e}}{\sqrt{\pi}} \frac{T_{\|e}}{T_{\perp e}} \left(\Delta_e - k^2 d_e^2\right), \qquad \text{where} \qquad \Delta_e \doteq \frac{T_{\perp e}}{T_{\|e}} - 1 > 0 \tag{16}$$

is the fractional temperature anisotropy and $d_e = c/\omega_{\mathrm{pe}}$ is the electron skin depth. Evidently, collisionless, unmagnetized, temperature-anisotropic plasmas can spontaneously grow magnetic fields at an exponential rate! This is the famous *Weibel instability*, which features prominently in several scenarios for cosmic magnetogenesis and in many laboratory laser-plasma experiments.

(f) Use equation (16) to show that the maximum growth rate of the Weibel instability and its associated wavenumber are given by

$$\gamma_{\max} \approx \frac{2\omega_{\mathrm{pe}}}{3\sqrt{3\pi}} \frac{v_{\mathrm{th}\|e}}{c} \frac{T_{\|e}}{T_{\perp e}} \Delta_e^{3/2} \qquad \text{and} \qquad k_{\max} d_e = (\Delta_e/3)^{1/2}, \tag{17}$$

respectively. Under what condition(s) is the asymptotic limit in which you obtained equation (16) a valid approximation for this solution?

(g) In order to grow, this instability requires particles to be able to free stream across the direction of the fluctuating magnetic field. With this knowledge, use equation (17) to give a rough estimate for the plasma β_e at which the instability should shut itself off. Explain your answer.