Introduction to Magnetohydrodynamic Turbulence

Ben Chandran, University of New Hampshire

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Goals - Similar to Yesterday's Talk

- To review a few things from earlier this week, as doing so can help key ideas to sink in.
- To 'de-mystify' the subject of MHD turbulence for you.
- To show you a few classic (and broadly useful) results, but also to explain carefully how you can recover those results for yourself.
- This means:
  - Wherever possible, I'm going to show you all the steps.
  - Much of this talk will be dry/mathematical, and there are many cool ideas and results that I will not have time to share with you (sorry).
  - But for many of you, this level of talk is not available elsewhere. Conference talks are way too advanced for students trying to learn about turbulence, and classes often don't cover this material. So I think you will find this useful, and worth your careful attention.

Outline

1. Quick review of magnetohydrodynamics (MHD)

2. Elsässer form of the incompressible MHD equations

3. Linear waves, weak turbulence, and strong turbulence

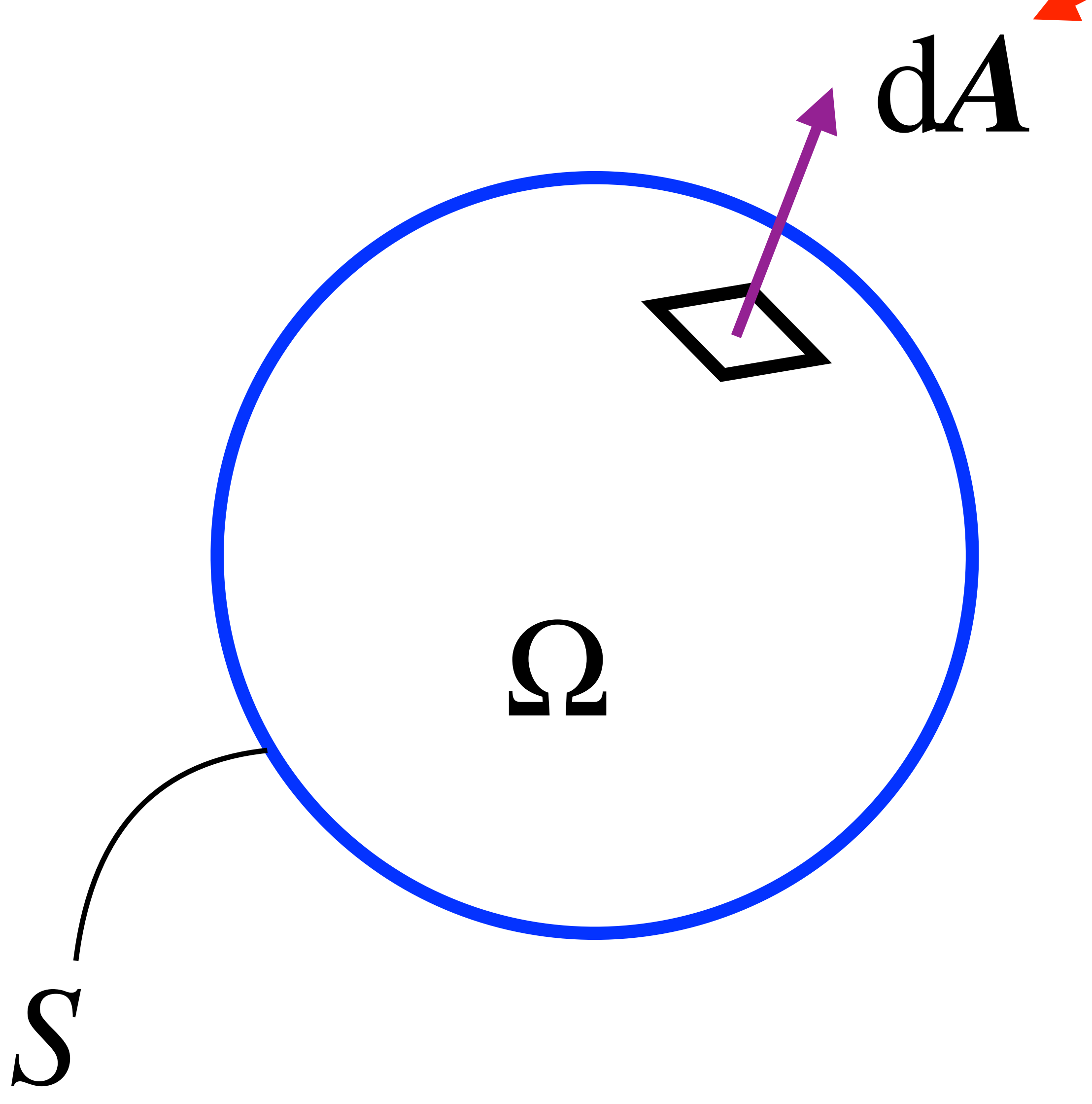
4. Weak incompressible MHD turbulence and the anisotropic energy cascade

5. Strong incompressible MHD turbulence and critical balance

6. Extras

Conservation of Mass

vectors in bold italic font



Consider an arbitrary fixed volume Ω with boundary S within a fluid with density $\rho(\boldsymbol{x}, t)$ and flow velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The mass within Ω is $M = \int_\Omega \rho \, \mathrm{d}^3 x$.

$\mathrm{d}M/\mathrm{d}t$ is just the rate at which mass flows in through the boundary of Ω

$$\longrightarrow \int_\Omega \frac{\partial \rho}{\partial t} \mathrm{d}^3 x = -\oint_S \rho \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} = -\int_\Omega \nabla \cdot (\rho \boldsymbol{u}) \, \mathrm{d}^3 x$$

(by Gauss's theorem)

As Ω is arbitrary,

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u})}$$

everywhere

'continuity equation'

Newton's Second Law: $\boldsymbol{a} = \boldsymbol{F}/m$

Suppose a fluid has velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. Is $\boldsymbol{a} = \frac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x}, t)$? No!

Consider a fluid element with position $\boldsymbol{x}(t)$ and velocity $\boldsymbol{u}(\boldsymbol{x}(t), t)$. Then

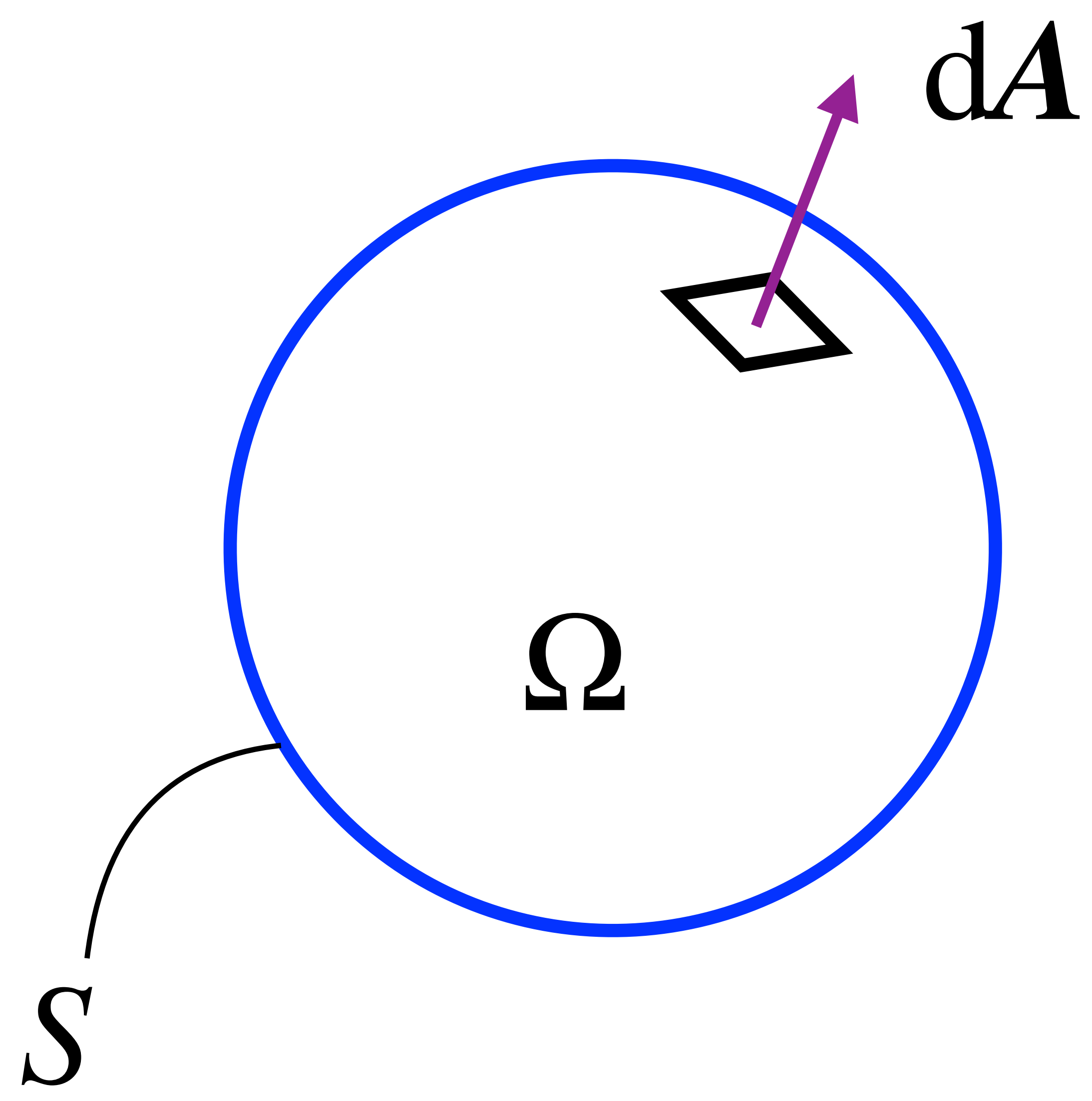
$$\boldsymbol{u}(\boldsymbol{x}(t), t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t),$$ and

$$\boldsymbol{a} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{x}(t), t) = \left(\frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \nabla\right)\boldsymbol{u}(\boldsymbol{x}(t), t) = \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)\boldsymbol{u}(\boldsymbol{x}(t), t)$$

The quantity $\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)$ is called the Lagrangian or convective time derivative.

It's the time derivative in a frame that follows the fluid.

Pressure Force Per Unit Volume $= -\nabla p$



Pressure force on an arbitrary fluid element of volume Ω with boundary S:

$$\boldsymbol{F} = -\oint_S p\,\mathrm{d}\boldsymbol{A} = -\oint_S p\boldsymbol{I}\cdot\mathrm{d}\boldsymbol{A} = -\int_\Omega \nabla\cdot(p\boldsymbol{I})\,\mathrm{d}^3x = -\int_\Omega \nabla p\,\mathrm{d}^3x$$

As Ω is arbitrary, the pressure force per unit volume everywhere is $-\nabla p$

($\boldsymbol{I}$ is the identity matrix. Third equality is Gauss's theorem applied to each component of $\boldsymbol{F}$ separately)

Lorentz Force Per Unit Volume

$$\sum_{\text{species s}} q_s n_s \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{u}_s \times \boldsymbol{B} \right)$$

q_s = charge of species s, n_s = number density of species s, $\boldsymbol{u}_{\text{rms}}$ = average velocity of species s

Lorentz Force Per Unit Volume

$$\sum_{\text{species s}} q_s n_s \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{u}_s \times \boldsymbol{B} \right) = \left(\sum_s q_s n_s \right) \boldsymbol{E} + \frac{1}{c} \left(\sum_s q_s n_s \boldsymbol{u}_s \right) \times \boldsymbol{B} = \frac{1}{c} \boldsymbol{J} \times \boldsymbol{B}$$

This is the charge density, which vanishes because plasma is quasineutral

This is the charge flux, which is by definition the current density $\boldsymbol{J}$

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Lorentz Force Per Unit Volume

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Ampere's Law: $\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J} - \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} \longrightarrow \boldsymbol{J} = \frac{c}{4\pi} \nabla \times \boldsymbol{B}$

In MHD, we drop this displacement-current term on the assumption that the characteristic phase velocities of fluctuations are much smaller than the speed of light.

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$$\left[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \right]_i = \epsilon_{ijk}(\epsilon_{jlm}\partial_l B_m)B_k = \epsilon_{jki}\epsilon_{jlm}B_k\partial_l B_m = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})B_k\partial_l B_m = (B_k\partial_k)B_i - \frac{1}{2}\partial_i(B_kB_k)$$

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$$\longrightarrow \sum_s q_s n_s \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}_s \times \boldsymbol{B} \right) = -\nabla \frac{B^2}{8\pi} + \frac{1}{4\pi} \boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

$$\text{force per unit mass} = \frac{\text{force per unit volume}}{\text{mass per unit volume}} = \frac{1}{\rho}\left[-\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}\right]$$

$$\longrightarrow \rho\left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}$$

Induction Equation (Ohm's Law)

For a fixed, metal conductor: $\boldsymbol{E} = \eta \boldsymbol{J}$, where $\boldsymbol{E}$ is the electric field and η is the resistivity

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For a perfectly conducting fluid with velocity $\boldsymbol{u}$, you replace $\boldsymbol{E}$ (the electric field in the lab frame) with $\boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \times \boldsymbol{B}$ (the electric frame in the plasma rest frame) and set $\eta = 0$

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Take curl of this equation and multiply by c: $\longrightarrow \quad c\nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0$

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Use Faraday's Law $c\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: $\longrightarrow \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$

Ideal MHD Equations

(Ideal means no dissipation — i.e., no resistivity or viscosity)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

$$\rho\left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B})$$

Is this a closed set of equations? I.e., can we solve them to determine the unknowns?

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$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

3 equations for the 4 variables $\rho, \boldsymbol{u}, \boldsymbol{B}, p \longrightarrow$ we need a 4th equation in order to solve for these 4 variables. This is the energy equation. We'll consider 3 simple examples.

Ideal Adiabatic MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \left(\frac{\partial}{\partial t} \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = - \nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

Adiabatic evolution: $$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \left(\frac{p}{\rho^\gamma} \right) = 0$$

(A reasonable approximation when the heat flux and other forms of heating/cooling can be neglected.)

Ideal Isothermal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi}\boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

$p = \rho c_s^2$, where the sound speed c_s is a constant

(A reasonable approximation when rapid heat conduction prevents much temperature variation.)

Ideal Incompressible MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho\left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\rho = \text{constant} \qquad \nabla \cdot \boldsymbol{u} = 0$$

(A reasonable approximation at large $\beta \equiv 8\pi p/B^2$ and small Mach number $u_{\rm rms}/c_{\rm s}$. But its simplicity has made it extremely useful for understanding MHD turbulence more generally, so we will focus on incompressible MHD turbulence today.)

Ideal Incompressible MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

Automatically satisfied given incompressibility conditions

$$\rho \left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi}\boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

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$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

$$\rho = \text{constant} \qquad \nabla \cdot \boldsymbol{u} = 0$$

L = correlation length or 'outer scale' of turbulence

Reynolds number $\mathrm{Re} = \frac{u_{\mathrm{rms}} L}{\nu}$ Magnetic Reynolds number $Re_{\mathrm{m}} = \frac{u_{\mathrm{rms}} L}{\eta}$

Key Ideas About MHD from Earlier Lectures

1. Two types of magnetic forces (per unit volume): the magnetic pressure force $-\nabla \dfrac{B^2}{8\pi}$ and the magnetic-tension force $\dfrac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}$

2. Flux conservation

3. Frozen-in law.

4. Alfvén waves.

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1. Quick review of magnetohydrodynamics (MHD)
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3. Linear waves, weak turbulence, and strong turbulence
4. Weak incompressible MHD turbulence and the anisotropic energy cascade
5. Strong incompressible MHD turbulence and critical balance
6. Extras: compressible turbulence, inverse cascade of magnetic helicity helicity barrier, cosmic-ray scattering by MHD turbulence

Change of Variables

$\frac{1}{\sqrt{4\pi\rho}}$ times the induction equation $\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$

$$\longrightarrow \quad \frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b}), \qquad \text{where } \boldsymbol{b} \equiv \frac{\boldsymbol{B}}{\sqrt{4\pi\rho}}.$$

$\frac{1}{\rho}$ times the momentum eq. $\rho\left(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B} \cdot \nabla \boldsymbol{B}$

$$\longrightarrow \quad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \quad \text{where} \quad \Pi \equiv \frac{p + (B^2/8\pi)}{\rho}$$

A Simpler Form for the Induction Equation

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}u_l b_m) = \epsilon_{kij}\epsilon_{klm}(b_m\partial_j u_l + u_l\partial_j b_m)$$

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$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = b_j\partial_j u_i + u_i\partial_j b_j - b_i\partial_j u_j - u_j\partial_j b_i$$

For example,

$$\delta_{il}\delta_{jm}b_m\partial_j u_l \leftrightarrow \sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\delta_{il}\delta_{jm}b_m\partial_j u_l$$

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$$\sum_{m=1}^{3}\delta_{mj}b_m = b_j$$ analogous to $$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

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$$\sum_{m=1}^{3}\delta_{mj}b_m=b_j$$

analogous to

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$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}\nabla \cdot \boldsymbol{b} - \boldsymbol{b}\nabla \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}$$

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$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}u_l b_m) = \epsilon_{kij}\epsilon_{klm}(b_m\partial_j u_l + u_l\partial_j b_m)$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(b_m\partial_j u_l + u_l\partial_j b_m)$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = b_j\partial_j u_i + u_i\partial_j b_j - b_i\partial_j u_j - u_j\partial_j b_i$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = [\boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}\nabla \cdot \boldsymbol{b} - \boldsymbol{b}\nabla \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}]_i$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \cancel{\boldsymbol{u}\nabla \cdot \boldsymbol{b}} - \cancel{\boldsymbol{b}\nabla \cdot \boldsymbol{u}} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}$$

A Simpler Form for the Induction Equation

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}u_l b_m) = \epsilon_{kij}\epsilon_{klm}(b_m\partial_j u_l + u_l\partial_j b_m)$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(b_m\partial_j u_l + u_l\partial_j b_m)$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = b_j\partial_j u_i + u_i\partial_j b_j - b_i\partial_j u_j - u_j\partial_j b_i$$

$$[\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i = [\boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}\nabla \cdot \boldsymbol{b} - \boldsymbol{b}\nabla \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}]_i$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}\cancel{\nabla \cdot \boldsymbol{b}} - \boldsymbol{b}\cancel{\nabla \cdot \boldsymbol{u}} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b}$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right)$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - \right.$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial \boldsymbol{z}^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) \cdot \nabla (\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) - (\boldsymbol{z}^{+} + \boldsymbol{z}^{-}) \cdot \nabla (\boldsymbol{z}^{+} + \boldsymbol{z}^{-})\right.$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial \boldsymbol{z}^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) \cdot \nabla (\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) - (\boldsymbol{z}^{+} + \boldsymbol{z}^{-}) \cdot \nabla (\boldsymbol{z}^{+} + \boldsymbol{z}^{-}) \pm (\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) \cdot \nabla (\boldsymbol{z}^{+} + \boldsymbol{z}^{-})\right.$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right) \qquad (1) \pm (2)\ \text{yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+} - z^{-})\cdot\nabla(z^{+} - z^{-}) - (z^{+} + z^{-})\cdot\nabla(z^{+} + z^{-}) \pm (z^{+} - z^{-})\cdot\nabla(z^{+} + z^{-}) \mp (z^{+} + z^{-})\cdot\nabla(z^{+} - z^{-})\right]$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

In case you're not used to this notation: you either take the upper sign in every $\pm$ and $\mp$, or you take the lower sign in every $\pm$ and $\mp$.

Now expand out all the products

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-}\right.$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + z^{-} \cdot \nabla z^{-} - z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - z^{-} \cdot \nabla z^{-}\right.$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2)\ \text{yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+} - z^{-})\cdot\nabla(z^{+} - z^{-}) - (z^{+} + z^{-})\cdot\nabla(z^{+} + z^{-}) \pm (z^{+} - z^{-})\cdot\nabla(z^{+} + z^{-}) \mp (z^{+} + z^{-})\cdot\nabla(z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-}$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\left.\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-} \mp z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \pm z^{-}\cdot\nabla z^{-}\right]$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[\cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + z^{-} \cdot \nabla z^{-} - \cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - z^{-} \cdot \nabla z^{-}\right.$$

$$\left. \pm z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \mp z^{-} \cdot \nabla z^{-} \mp z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \pm z^{-} \cdot \nabla z^{-}\right]$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right) \qquad \text{(1)} \pm \text{(2) yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\Big[\cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + \cancel{z^{-} \cdot \nabla z^{-}} - \cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - \cancel{z^{-} \cdot \nabla z^{-}}$$

$$\pm z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \mp z^{-} \cdot \nabla z^{-} \mp z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \pm z^{-} \cdot \nabla z^{-}\Big]$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right)$$

(1) ± (2) yields:

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[\cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + \cancel{z^{-} \cdot \nabla z^{-}} - \cancel{z^{+} \cdot \nabla z^{+}} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - \cancel{z^{-} \cdot \nabla z^{-}}\right.$$

$$\left. \cancel{\pm z^{+} \cdot \nabla z^{+}} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \mp z^{-} \cdot \nabla z^{-} \cancel{\mp z^{+} \cdot \nabla z^{+}} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \pm z^{-} \cdot \nabla z^{-}\right]$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial \boldsymbol{z}^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(\boldsymbol{z}^{+} - \boldsymbol{z}^{-})\cdot\nabla(\boldsymbol{z}^{+} - \boldsymbol{z}^{-}) - (\boldsymbol{z}^{+} + \boldsymbol{z}^{-})\cdot\nabla(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}) \pm (\boldsymbol{z}^{+} - \boldsymbol{z}^{-})\cdot\nabla(\boldsymbol{z}^{+} + \boldsymbol{z}^{-}) \mp (\boldsymbol{z}^{+} + \boldsymbol{z}^{-})\cdot\nabla(\boldsymbol{z}^{+} - \boldsymbol{z}^{-})\right]$$

$$\frac{\partial}{\partial t}\boldsymbol{z}^{\pm} = -\nabla\Pi + \frac{1}{4}\left[\cancel{\boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{+}} - \boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{-} - \boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{+} + \cancel{\boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{-}} - \cancel{\boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{+}} - \boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{-} - \boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{+} - \cancel{\boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{-}}\right.$$

$$\left.\pm\cancel{\boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{+}} \pm \boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{-} \mp \boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{+} \mp \cancel{\boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{-}} \mp \cancel{\boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{+}} \pm \boldsymbol{z}^{+}\cdot\nabla\boldsymbol{z}^{-} \mp \boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{+} \pm \cancel{\boldsymbol{z}^{-}\cdot\nabla\boldsymbol{z}^{-}}\right]$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\left.\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-} \mp z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \pm z^{-}\cdot\nabla z^{-}\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{2}\left[-z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+}\right]$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + z^{-} \cdot \nabla z^{-} - z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - z^{-} \cdot \nabla z^{-}\right.$$

$$\left. \pm z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \mp z^{-} \cdot \nabla z^{-} \mp z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \pm z^{-} \cdot \nabla z^{-}\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{2}\left[-z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+}\right]$$

Choose the upper sign in each $\pm$ and $\mp$: $\longrightarrow \dfrac{\partial z^{+}}{\partial t} = -\nabla \Pi - z^{-} \cdot \nabla z^{+}$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+} - z^{-}) \cdot \nabla (z^{+} - z^{-}) - (z^{+} + z^{-}) \cdot \nabla (z^{+} + z^{-}) \pm (z^{+} - z^{-}) \cdot \nabla (z^{+} + z^{-}) \mp (z^{+} + z^{-}) \cdot \nabla (z^{+} - z^{-})\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} + z^{-} \cdot \nabla z^{-} - z^{+} \cdot \nabla z^{+} - z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} - z^{-} \cdot \nabla z^{-}\right.$$

$$\left. \pm z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \mp z^{-} \cdot \nabla z^{-} \mp z^{+} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+} \pm z^{-} \cdot \nabla z^{-}\right]$$

$$\frac{\partial}{\partial t} z^{\pm} = -\nabla \Pi + \frac{1}{2}\left[-z^{+} \cdot \nabla z^{-} - z^{-} \cdot \nabla z^{+} \pm z^{+} \cdot \nabla z^{-} \mp z^{-} \cdot \nabla z^{+}\right]$$

Choose the lower sign in each $\pm$ and $\mp$: $\longrightarrow \dfrac{\partial z^{-}}{\partial t} = -\nabla \Pi - z^{+} \cdot \nabla z^{-}$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\left.\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-} \mp z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \pm z^{-}\cdot\nabla z^{-}\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{2}\left[-z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+}\right]$$

Both cases can thus be represented via

$$\boxed{\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi - z^{\mp}\cdot\nabla z^{\pm}}$$

Change to Elsässer Variables $z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \Pi + \boldsymbol{b}\cdot\nabla \boldsymbol{b} - \boldsymbol{u}\cdot\nabla \boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla \boldsymbol{u} - \boldsymbol{u}\cdot\nabla \boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla \Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla \Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\left.\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-} \mp z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \pm z^{-}\cdot\nabla z^{-}\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla \Pi + \frac{1}{2}\left[-z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+}\right]$$

$$\boxed{\frac{\partial}{\partial t}z^{\pm} = -\nabla \Pi - z^{\mp}\cdot\nabla z^{\pm}}$$

Change to Elsässer Variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\Pi + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (1) \qquad\qquad \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{b} \qquad (2)$$

$$z^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} \qquad \boldsymbol{u} = \frac{1}{2}\left(z^{+} + z^{-}\right) \qquad \boldsymbol{b} = \frac{1}{2}\left(z^{+} - z^{-}\right) \qquad (1) \pm (2) \text{ yields:}$$

$$\frac{\partial z^{\pm}}{\partial t} = -\nabla\Pi + \frac{1}{4}\left[(z^{+}-z^{-})\cdot\nabla(z^{+}-z^{-}) - (z^{+}+z^{-})\cdot\nabla(z^{+}+z^{-}) \pm (z^{+}-z^{-})\cdot\nabla(z^{+}+z^{-}) \mp (z^{+}+z^{-})\cdot\nabla(z^{+}-z^{-})\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{4}\left[z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} + z^{-}\cdot\nabla z^{-} - z^{+}\cdot\nabla z^{+} - z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} - z^{-}\cdot\nabla z^{-}\right.$$

$$\left.\pm z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \mp z^{-}\cdot\nabla z^{-} \mp z^{+}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+} \pm z^{-}\cdot\nabla z^{-}\right]$$

$$\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi + \frac{1}{2}\left[-z^{+}\cdot\nabla z^{-} - z^{-}\cdot\nabla z^{+} \pm z^{+}\cdot\nabla z^{-} \mp z^{-}\cdot\nabla z^{+}\right]$$

$$\boxed{\frac{\partial}{\partial t}z^{\pm} = -\nabla\Pi - z^{\mp}\cdot\nabla z^{\pm}}$$

Also, $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = \nabla\cdot z^{\pm} = 0$

Alternative Formulation of Elsässer Variables

Let $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, where $\boldsymbol{B}_0$ is the mean magnetic field, which is a constant.

Then $\boldsymbol{b} = \dfrac{\boldsymbol{B}}{\sqrt{4\pi\rho}} = \boldsymbol{v}_{\mathrm{A}} + \delta\boldsymbol{b}$, where $\boldsymbol{v}_{\mathrm{A}} = \dfrac{\boldsymbol{B}_0}{\sqrt{4\pi\rho}}$ is the Alfvén velocity,

and $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b} = \boldsymbol{u} \pm \delta\boldsymbol{b} \pm \boldsymbol{v}_{\mathrm{A}} \equiv \boldsymbol{w}^{\pm} \pm \boldsymbol{v}_{\mathrm{A}}$, where $\boldsymbol{w}^{\pm} \equiv \boldsymbol{u} \pm \delta\boldsymbol{b}$

Substitute this expression into $\dfrac{\partial \boldsymbol{z}^{\pm}}{\partial t} = -\nabla\Pi - \boldsymbol{z}^{\mp}\cdot\nabla\boldsymbol{z}^{\pm}$. Note that $\boldsymbol{v}_{\mathrm{A}}$ is a constant.

$$\longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{w}^{\pm} = -\nabla\Pi - \boldsymbol{w}^{\mp}\cdot\nabla\boldsymbol{w}^{\pm} \pm \boldsymbol{v}_{\mathrm{A}}\cdot\nabla\boldsymbol{w}^{\pm}$$

$$\longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}}\cdot\nabla\boldsymbol{w}^{\pm} = -\nabla\Pi - \boldsymbol{w}^{\mp}\cdot\nabla\boldsymbol{w}^{\pm}$$

Important Slide: A Few Key Points About the Elsässer Form of MHD

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

- As in our discussion of hydrodynamic turbulence yesterday, the role of the pressure term $-\nabla \Pi$ is simply to cancel out the compressive part of the nonlinear term $-\boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$ to maintain $\nabla \cdot \boldsymbol{w}^{\pm} = 0$.

- As $w^{\pm}/v_{\mathrm{A}} \rightarrow 0$, the right-hand side of Eq. (1) becomes negligible $\longrightarrow$ $\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = 0.$

  The solution to this linear advection equation is $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x} \pm \boldsymbol{v}_{\mathbf{A}} t)$, where $\boldsymbol{f}$ is an arbitrary function.

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$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\rm A} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

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$$\frac{\partial}{\partial t} f(x \pm v_{{\rm A}x}t, y \pm v_{{\rm A}y}t, z \pm v_{{\rm A}z}t) = \pm v_{{\rm A}x}\frac{\partial f}{\partial x} \pm v_{{\rm A}y}\frac{\partial f}{\partial y} \pm v_{{\rm A}z}\frac{\partial f}{\partial z} = \pm \boldsymbol{v}_{\rm A} \cdot \nabla f$$

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$$\frac{\partial}{\partial t} f(x \pm v_{\mathrm{A}x}t, y \pm v_{\mathrm{A}y}t, z \pm v_{\mathrm{A}z}t) = \pm v_{\mathrm{A}x}\frac{\partial f}{\partial x} \pm v_{\mathrm{A}y}\frac{\partial f}{\partial y} \pm v_{\mathrm{A}z}\frac{\partial f}{\partial z} = \pm \boldsymbol{v}_{\mathrm{A}} \cdot \nabla f$$

$$\longrightarrow \frac{\partial}{\partial t} f(\boldsymbol{x} \pm \boldsymbol{v}_{\mathrm{A}}t) \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla f(\boldsymbol{x} \pm \boldsymbol{v}_{\mathrm{A}}t) = 0$$

Important Slide: A Few Key Points About the Elsässer Form of MHD

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  The solution to this linear advection equation is $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x} \pm \boldsymbol{v}_{\rm A} t)$, where $\boldsymbol{f}$ is an arbitrary function. This solution describes $\boldsymbol{w}^{\pm}$ 'fluctuations' that propagate at velocity $\mp \boldsymbol{v}_{\rm A}$. These are linear Alfvén waves and (the high-β limit of) slow magnetosonic waves (sometimes referred to as pseudo-Alfvén waves). Note: $\boldsymbol{w}^{+}$ propagates anti-parallel to $\boldsymbol{B}_0$, and $\boldsymbol{w}^{-}$ propagates parallel to $\boldsymbol{B}_0$.

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- If either $\boldsymbol{w}^{+}$ or $\boldsymbol{w}^{-}$ vanishes throughout an open region of space, then the nonlinear term vanishes throughout that region. $\longrightarrow$ nonlinear interactions (and hence turbulence) arises only from 'collisions' between counter-propagating waves. (Iroshnikov 1963; Kraichnan 1965)

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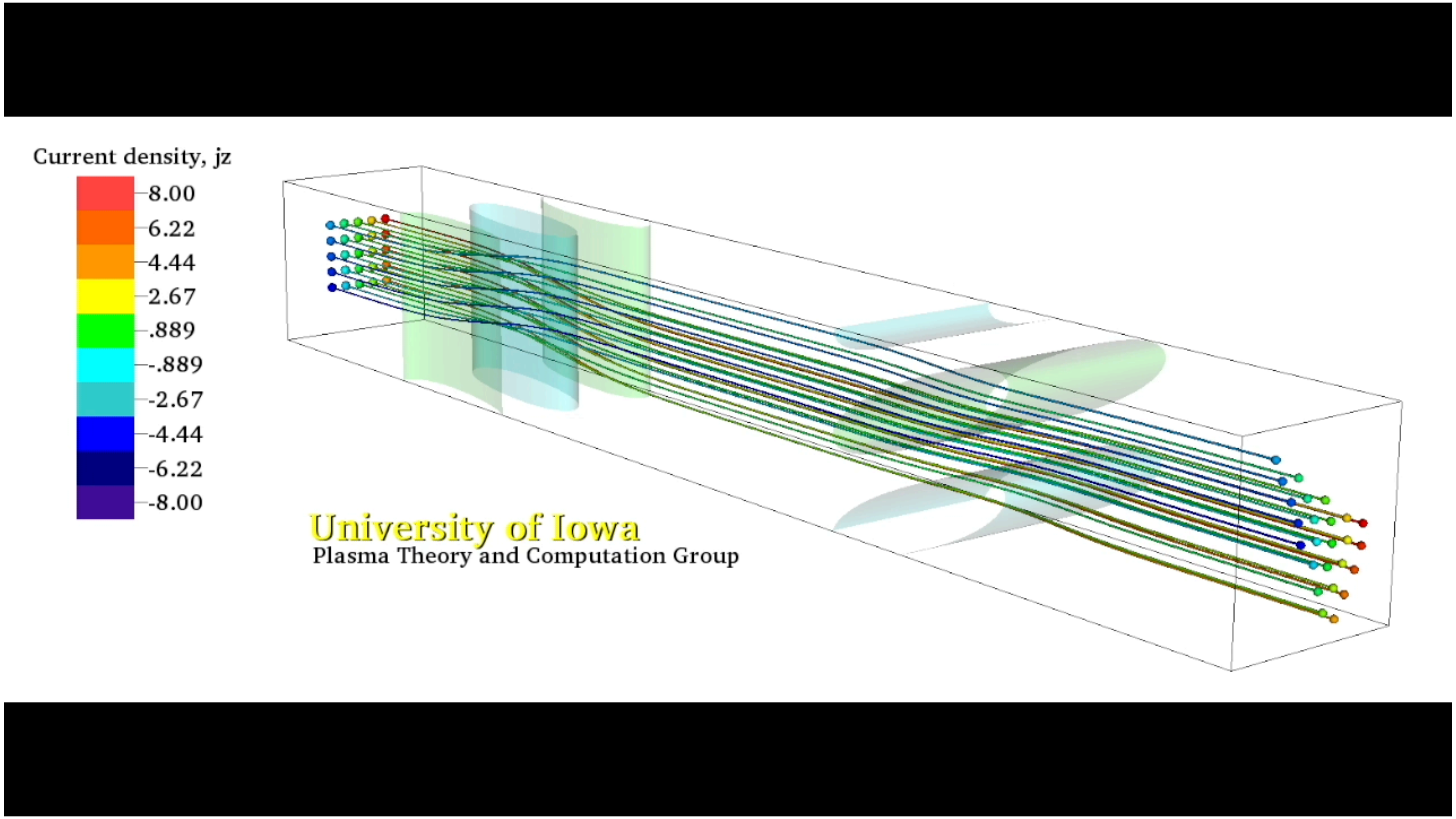
- If either $\boldsymbol{w}^{+}$ or $\boldsymbol{w}^{-}$ vanishes throughout an open region of space, then the nonlinear term vanishes throughout that region. $\longrightarrow$ nonlinear interactions (and hence turbulence) arises only from 'collisions' between counter-propagating waves. (Iroshnikov 1963; Kraichnan 1965)

- The linear solution $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x} \pm \boldsymbol{v}_{\rm A} \boldsymbol{t})$ is an exact nonlinear solution if $\boldsymbol{w}^{\mp} = 0$.

Movie of Colliding Alfvén-Wave Packets



Howes, Verniero, & Klein (2016)

Conservation Laws

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

Take dot product with $\boldsymbol{w}^{\pm}$ $\longrightarrow$ $\frac{1}{2}\frac{\partial}{\partial t}\left(w^{\pm}\right)^2 + \frac{1}{2}\boldsymbol{v}_{\mathrm{A}} \cdot \nabla \left(w^{\pm}\right)^2 = -\boldsymbol{w}^{\pm} \cdot \nabla \Pi - \frac{1}{2}\boldsymbol{w}^{\mp} \cdot \nabla \left(w^{\pm}\right)^2$

Use $\nabla \cdot \boldsymbol{w}^{\pm} = \nabla \cdot \boldsymbol{v}_{\mathrm{A}} = 0$ $\longrightarrow$ $\frac{1}{2}\frac{\partial}{\partial t}\left(w^{\pm}\right)^2 + \frac{1}{2}\nabla \cdot \left(\boldsymbol{v}_{\mathrm{A}} w^{\pm}\right)^2 = -\nabla \cdot \left(\boldsymbol{w}^{\pm}\Pi\right) - \frac{1}{2}\nabla \cdot \left(\boldsymbol{w}^{\mp} w^{\pm}\right)^2$

Integrate over all of space. The pure divergence terms become, via Gauss's theorem, surface integrals at infinite, which vanish because the plasma is confined to a finite volume. We then obtain

$\frac{\mathrm{d}\mathscr{E}^{\pm}}{\mathrm{d}t} = 0$ where $\mathscr{E}^{\pm} \equiv \frac{1}{4}\int_{\text{all space}} \mathrm{d}^3x \left(w^{\pm}\right)^2$ is the energy per unit mass in $w^{\pm}$ fluctuations.

KEY POINT: because $\mathscr{E}^{+}$ and $\mathscr{E}^{-}$ are separately conserved, nonlinear interactions cannot transfer energy from w^{+} to w^{-}, or vice versa. However, nonlinear interactions can transfer energy between scales, and both energy and cross helicity cascade from large scales to small scales.

Cross Helicity

The cross helicity is defined as $\mathscr{H}_c = \mathscr{E}^+ - \mathscr{E}^- = \dfrac{1}{4}\displaystyle\int_{\text{all space}} \mathrm{d}^3x \left[\left(w^+\right)^2 - (w^-)^2\right]$

$$= \frac{1}{4}\int_{\text{all space}} \mathrm{d}^3x \left(u^2 + 2\boldsymbol{u}\cdot\boldsymbol{b} + b^2 - u^2 + 2\boldsymbol{u}\cdot\boldsymbol{b} - b^2\right)$$

$$= \int_{\text{all space}} \mathrm{d}^3x\, \boldsymbol{u}\cdot\boldsymbol{b}$$

Because $\mathscr{E}^+$ and $\mathscr{E}^-$ are separately conserved, the cross helicity and total energy $\mathscr{E} = \mathscr{E}^+ + \mathscr{E}^-$ are both conserved.

We will consider ‘balanced turbulence’ in which $\mathscr{H}_c = 0$, but ‘imbalanced turbulence’ with nonzero $\mathscr{H}_c$ plays an important role in systems like the solar wind.

Outline

1. Quick review of magnetohydrodynamics (MHD)

2. Elsässer form of the incompressible MHD equations

3. Linear waves, weak turbulence, and strong turbulence

4. Weak incompressible MHD turbulence and the anisotropic energy cascade

5. Strong incompressible MHD turbulence and critical balance

6. Extras: compressible turbulence, inverse cascade of magnetic helicity helicity barrier, cosmic-ray scattering by MHD turbulence

Three Regimes of Waves and Turbulence.

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

1. Linear waves. Ignore the nonlinear term entirely. The pressure fluctuation is then negligible (see discussion of the pressure term in yesterday's lecture), and we recover just linear waves: Alfvén waves and the incompressible (high-β) limit of the slow magnetosonic wave.

2. Weak turbulence. You keep the nonlinear $\boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$ term, but treat it as small compared to the linear $\boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm}$ term. In this case, the 'zeroth order' solution to equation (1) is a bunch of linear waves, and then the higher-order solutions to this equation allow for interactions between these waves. Waves will oscillate many times at their linear frequencies before being distorted appreciably by nonlinear interactions.

3. Strong turbulence. The nonlinear $\boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$ term is comparable to or much larger than the linear $\boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm}$ term. This regime is analogous to hydrodynamic turbulence or a critically damped harmonic oscillator. Waves will undergo $\lesssim 1$ oscillation before being strongly distorted by nonlinear interactions.

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Magnetic-Field-Line Displacements

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

Magnetic-Field-Line Displacements

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

$$\longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{w}^{\pm} + \left(\boldsymbol{w}^{\mp} \mp \boldsymbol{v}_{\mathrm{A}}\right) \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi \qquad (2)$$

Magnetic-Field-Line Displacements

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If you ignore the $-\nabla \Pi$ term, Equation (2) is an advection equation for $\boldsymbol{w}^{\pm}$, which states that $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t)$ is advected at the velocity $\boldsymbol{w}^{\mp} \mp \boldsymbol{v}_{\mathrm{A}}$.

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If you ignore the $-\nabla \Pi$ term, Equation (2) is an advection equation for $\boldsymbol{w}^{\pm}$, which states that $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t)$ is advected at the velocity $\boldsymbol{w}^{\mp} \mp \boldsymbol{v}_{\mathrm{A}}$.

If $\boldsymbol{w}^{\mp} = 0$, then $\boldsymbol{w}^{\pm}$ is advected at velocity $\mp \boldsymbol{v}_{\mathrm{A}}$ along the field lines of the background magnetic field $\boldsymbol{B}_0$.

Magnetic-Field-Line Displacements

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm} \qquad (1)$$

$$\longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{w}^{\pm} + \left(\boldsymbol{w}^{\mp} \mp \boldsymbol{v}_{\mathrm{A}}\right) \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla \Pi \qquad (2)$$

If you ignore the $-\nabla\Pi$ term, Equation (2) is an advection equation for $\boldsymbol{w}^{\pm}$, which states that $\boldsymbol{w}^{\pm}(\boldsymbol{x}, t)$ is advected at the velocity $\boldsymbol{w}^{\mp} \mp \boldsymbol{v}_{\mathrm{A}}$.

If $\boldsymbol{w}^{\mp} = 0$, then $\boldsymbol{w}^{\pm}$ is advected at velocity $\mp\boldsymbol{v}_{\mathrm{A}}$ along the field lines of the background magnetic field $\boldsymbol{B}_0$.

If $0 < w^{\mp} \ll v_{\mathrm{A}}$, then $\boldsymbol{w}^{\pm}$ is advected along the fields lines of the sum of $\boldsymbol{B}_0$ and the part of $\delta\boldsymbol{B}$ that arises from $\boldsymbol{w}^{\mp}$ (Maron & Goldreich 2001)

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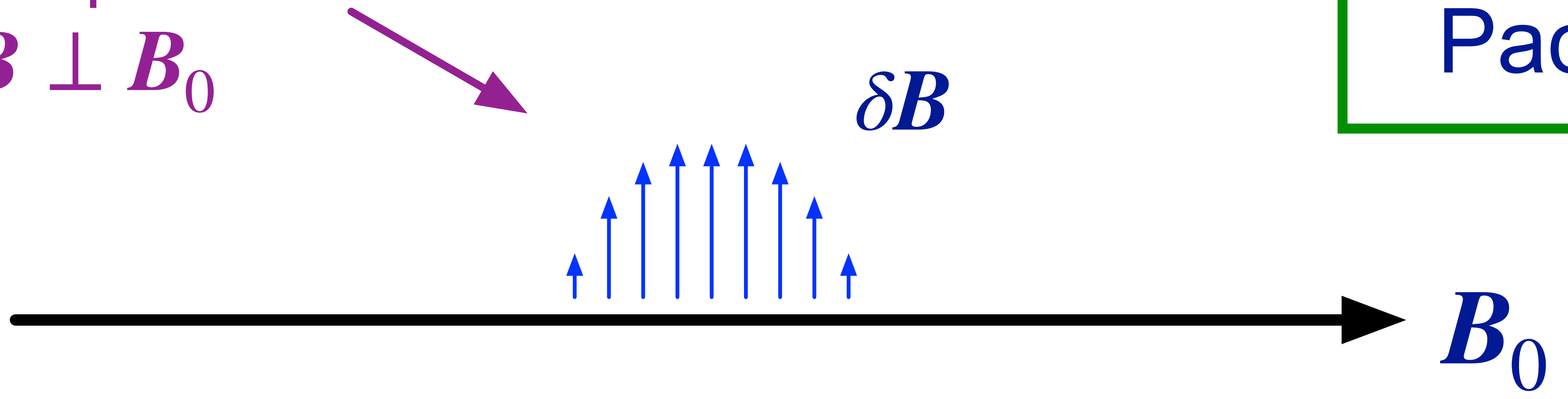
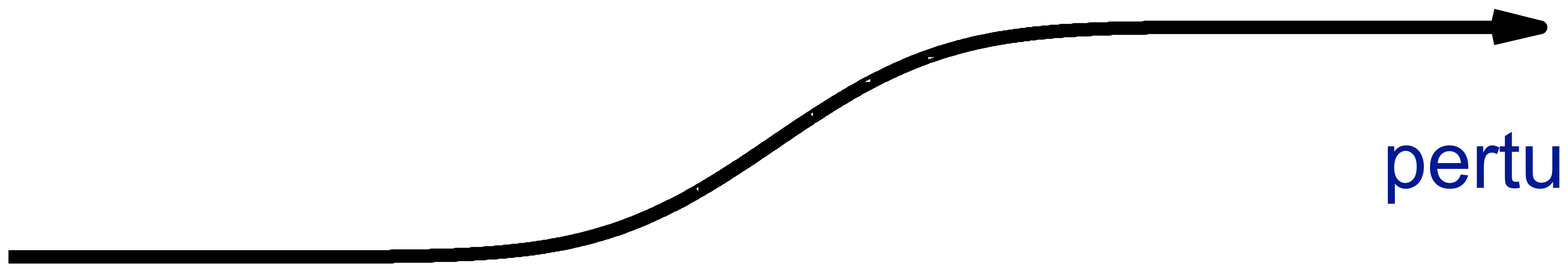
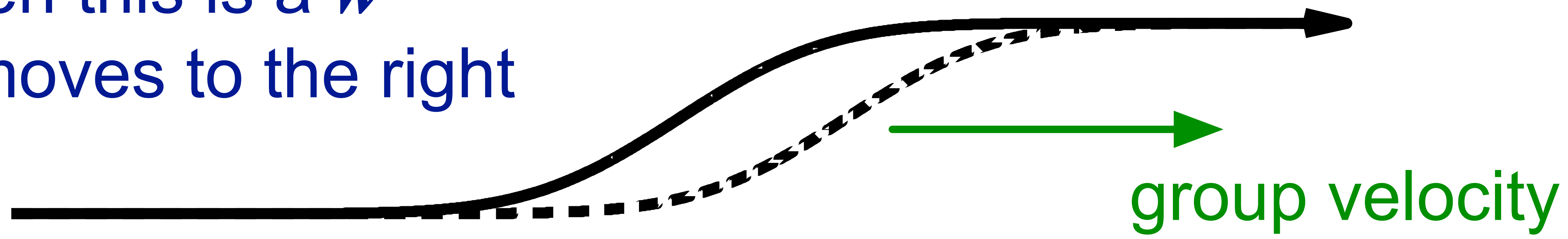
The way that $\boldsymbol{w}^{+}$ and $\boldsymbol{w}^{-}$ displace magnetic-field lines is the key to understanding nonlinear wave-wave interactions.

An Alfven Wave Packet in 1D



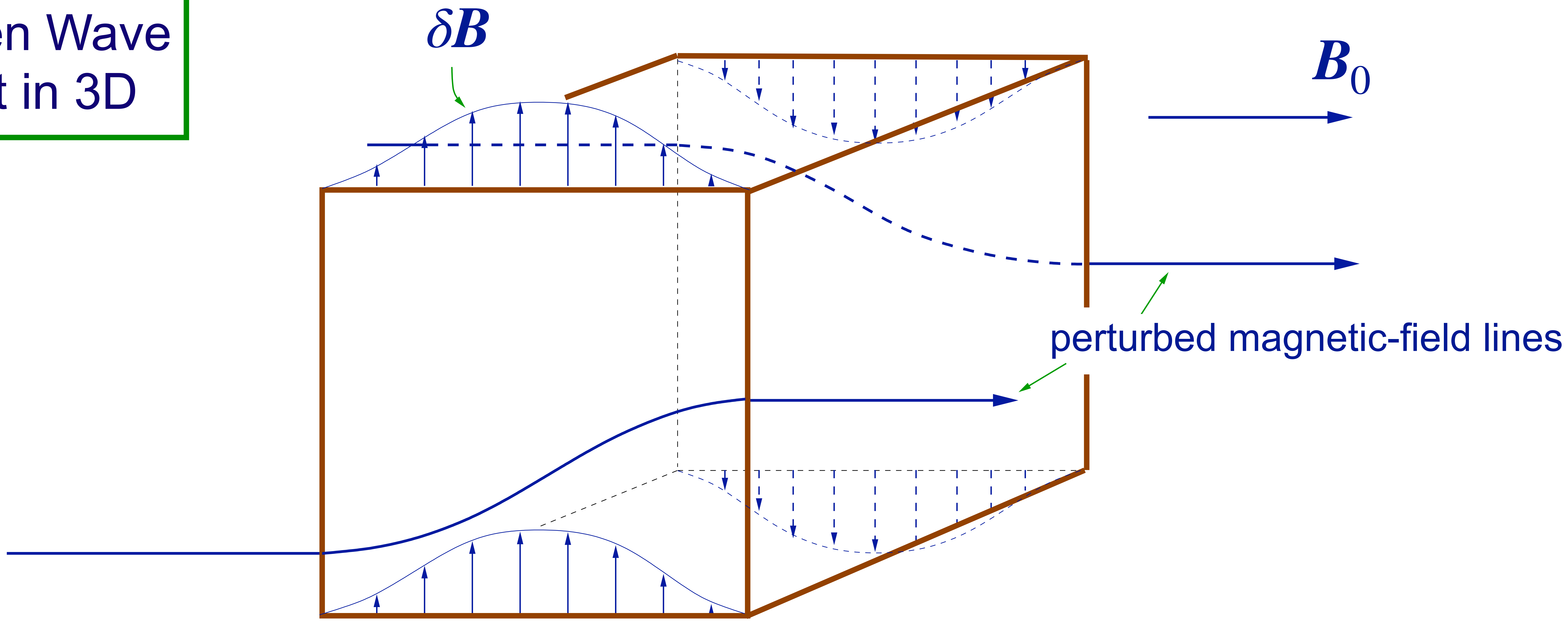


if $\boldsymbol{u} = -\delta \boldsymbol{b}$, then this is a $\boldsymbol{w}^-$ wave packet that moves to the right



An 'incoming' $\boldsymbol{w}^+$ wave packet from the right would follow the perturbed field line, moving to the left and down.

An Alfven Wave Packet in 3D

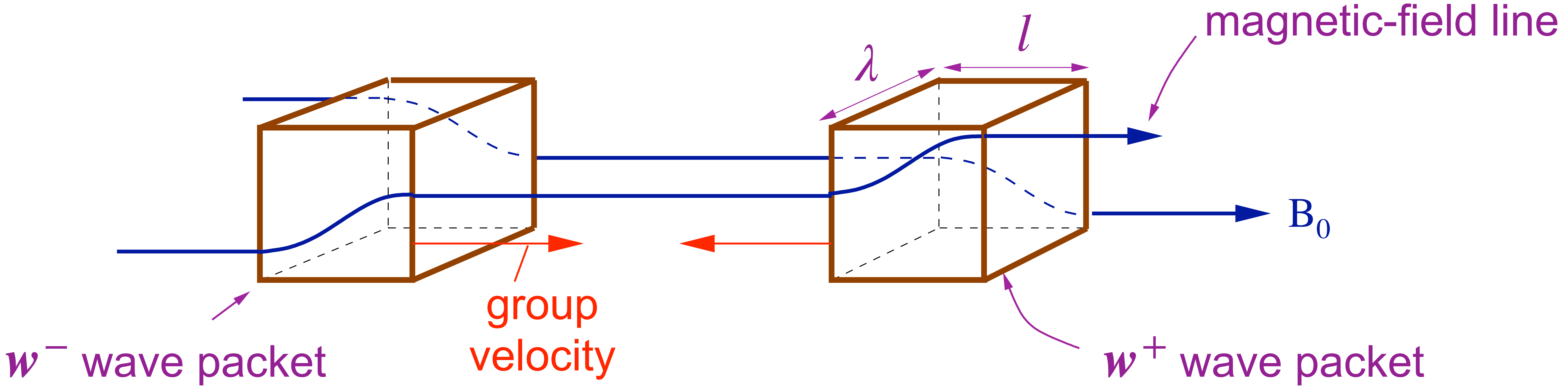


If $\boldsymbol{u} = -\,\delta\boldsymbol{b}$, then $\boldsymbol{w}^+ = 0$ and this is a $\boldsymbol{w}^-$ wave packet that propagates to the right without distortion.

An 'incoming' $\boldsymbol{w}^+$ wave packet from the right would follow the perturbed magnetic field lines, moving left and down in the plane of the cube nearest you and moving to the left and up in the plane of the cube farthest from you.

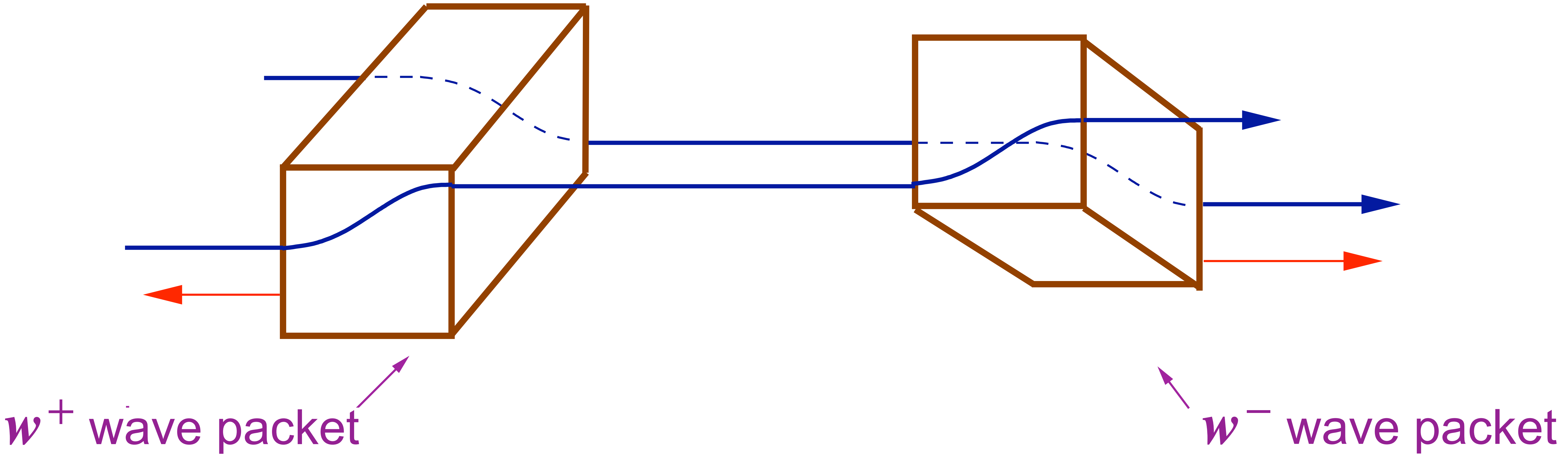
A Wave Packet Collision

BEFORE COLLISION:



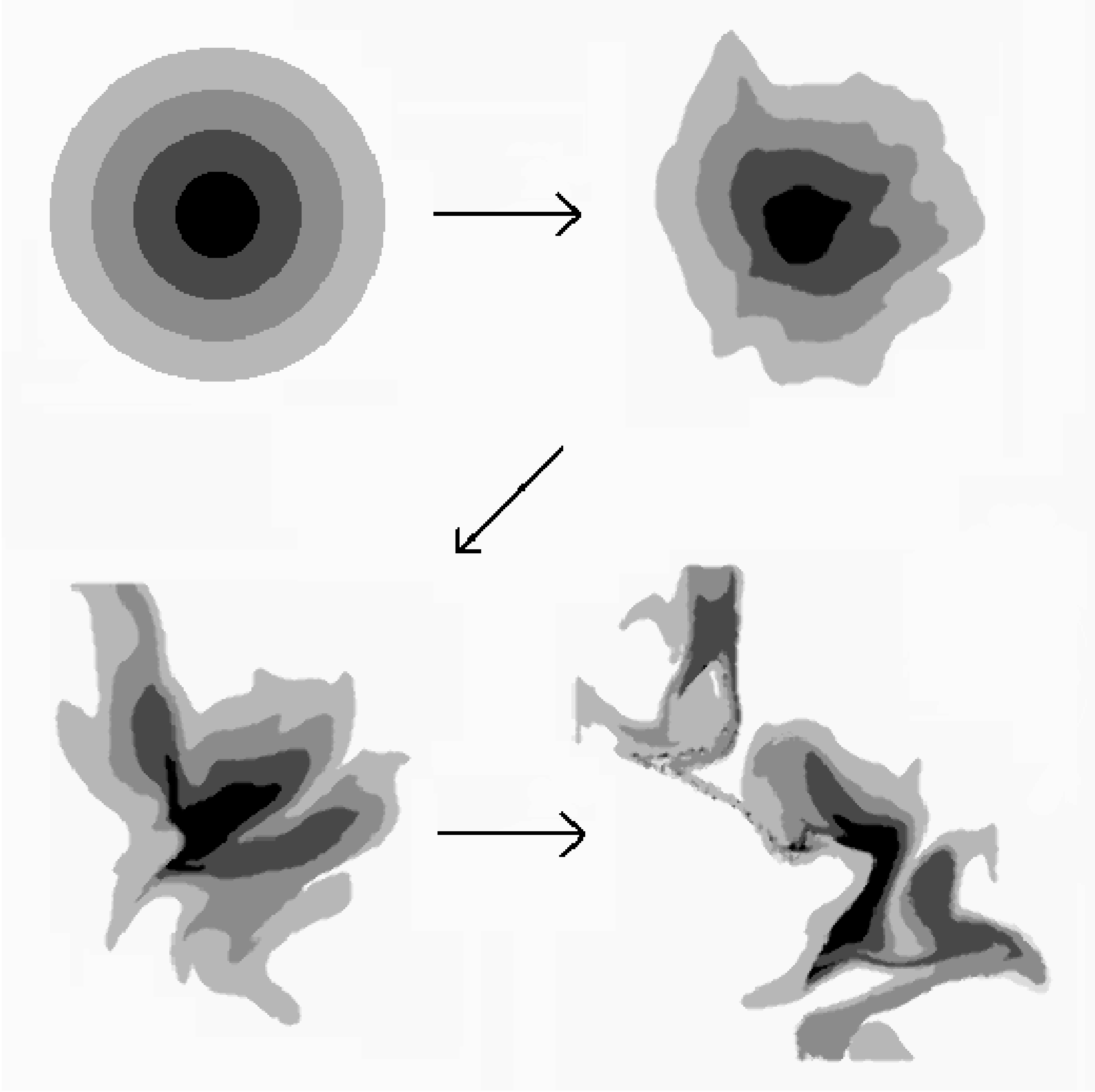
DURING COLLISION: each wave packet follows the field lines of the other wave packet

AFTER COLLISION: wave packets have passed through each other and have been sheared



This shearing reduces the perpendicular length scale λ of the wave packets

Shearing of a wave packet by field-line wandering



Maron & Goldreich (2001)

As wave packets follow the perturbed field lines in a turbulent plasma, their perpendicular correlation lengths get smaller and smaller. This gives rise to the same type of energy cascade that we saw yesterday in our discussion of hydrodynamic turbulence.

Anisotropic Energy Cascade

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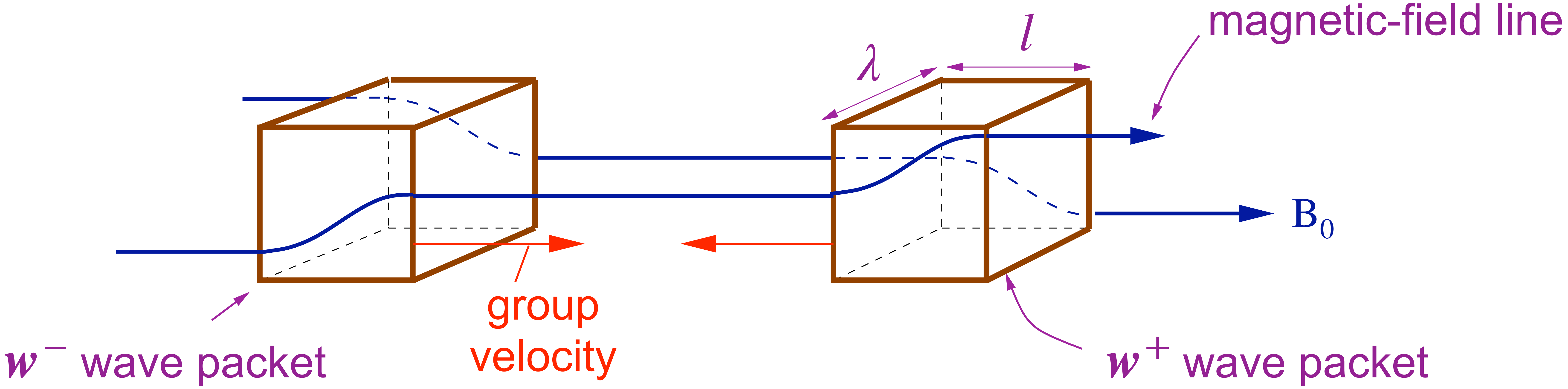

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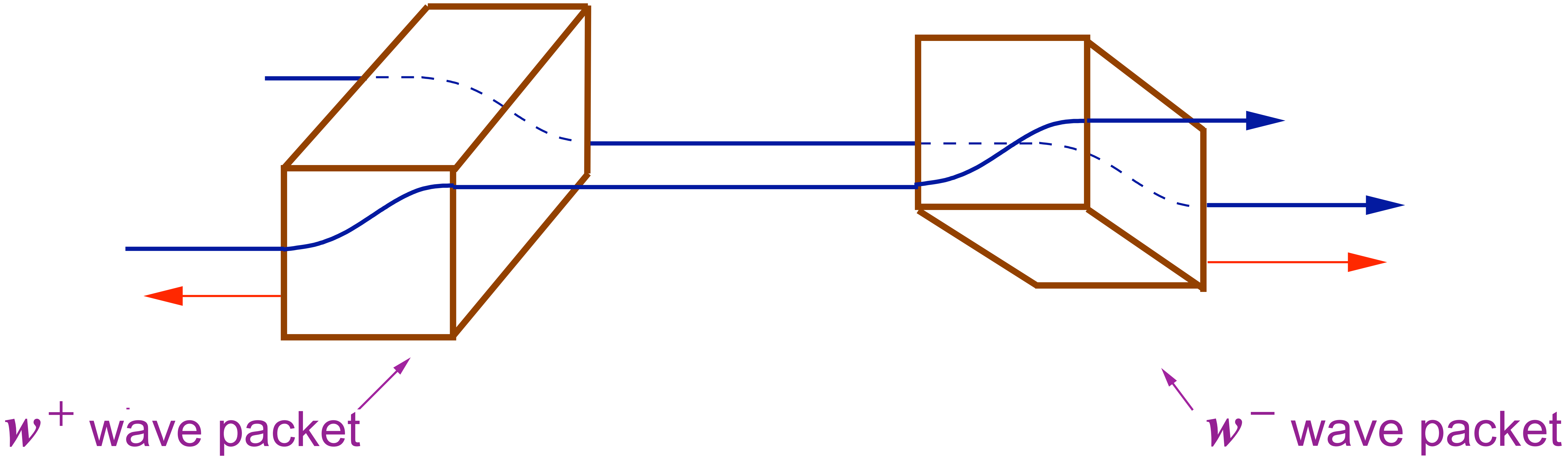

In weak turbulence, neither wave packet is changed appreciably during a single ‘collision,’ so, e.g., the right and left sides of the ‘incoming’ w^+ wave packet are affected in almost exactly the same way by the collision. This means that the collision does not alter the structure of the wave packet along the field line. You thus get small-scale structure transverse to the magnetic field, but not along the magnetic field. I.e., you get small λ, but not small l. (Shebalin et al 1983, Ng & Bhattacharejee 1997, Goldreich & Sridhar 1997).

Resonant 3-Wave Interactions in Weak Incompressible MHD Turbulence

(Shebalin et al 1983)

Wavenumber matching condition: $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ (1)

Frequency matching condition:

$$\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}} \qquad (2)$$

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$$\longrightarrow \omega_{\boldsymbol{k}} = +k_z v_{\mathrm{A}} \qquad \omega_{\boldsymbol{p}} = +p_z v_{\mathrm{a}} \qquad \omega_{\boldsymbol{q}} = -q_z v_{\mathrm{A}}$$

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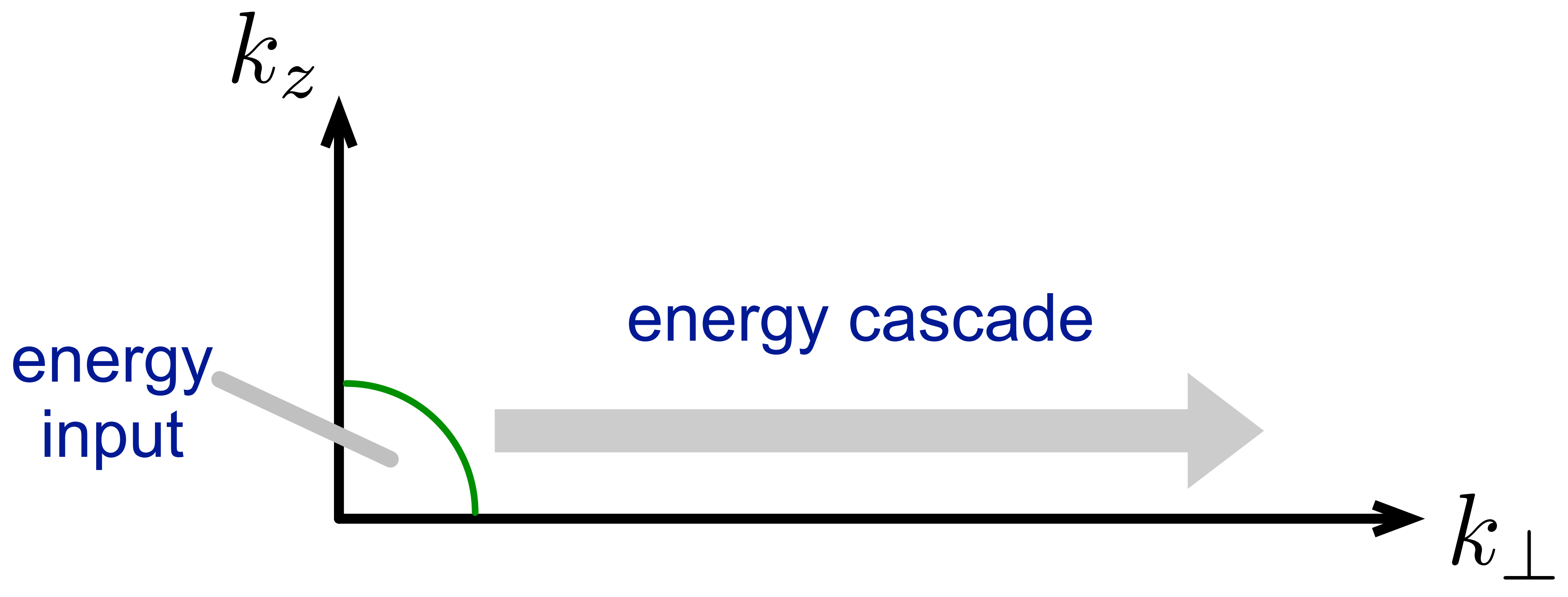
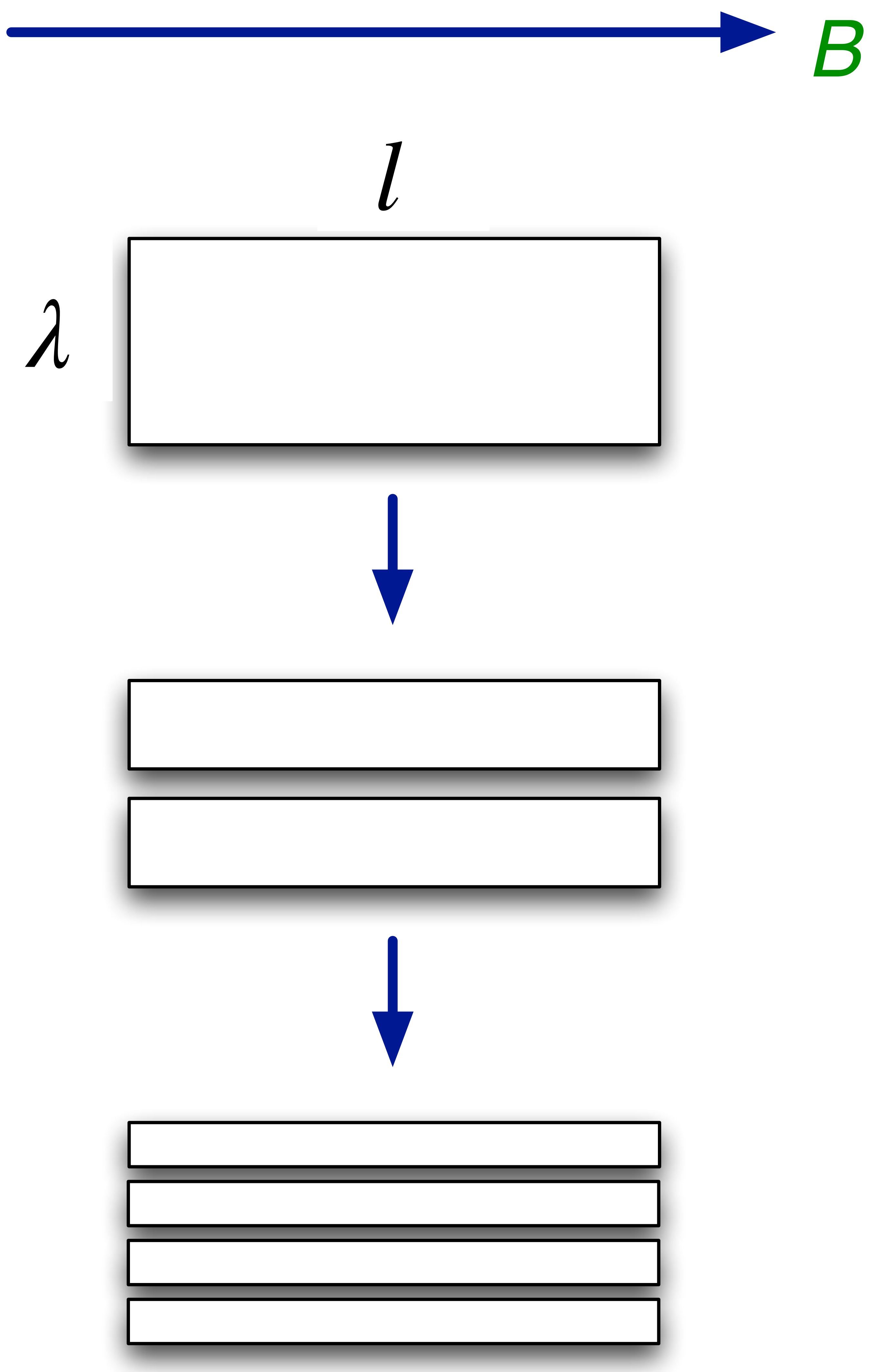


(No way to transfer energy to higher k_z)

Anisotropic Cascade in Weak MHD Turbulence

(Shebalin, Montgomery, & Matthaeus 1983)



- As energy cascades to smaller scales, you can think of wave packets breaking up into smaller wave packets.

- During this process, λ decreases, but l does not.

- Fluctuations with small λ end up being very anisotropic, with $\lambda \ll l$.

- In wavenumber $(\boldsymbol{k})$ space, most of the energy at large wavenumbers is in the region where $k_\perp \gg k_\parallel$, where $k_\perp$ $\left(k_\parallel\right)$ is the component of $\boldsymbol{k}$ perpendicular (parallel) to the background magnetic field.

(Goldreich & Sridhar 1997, Ng & Bhattacharjee 1997)

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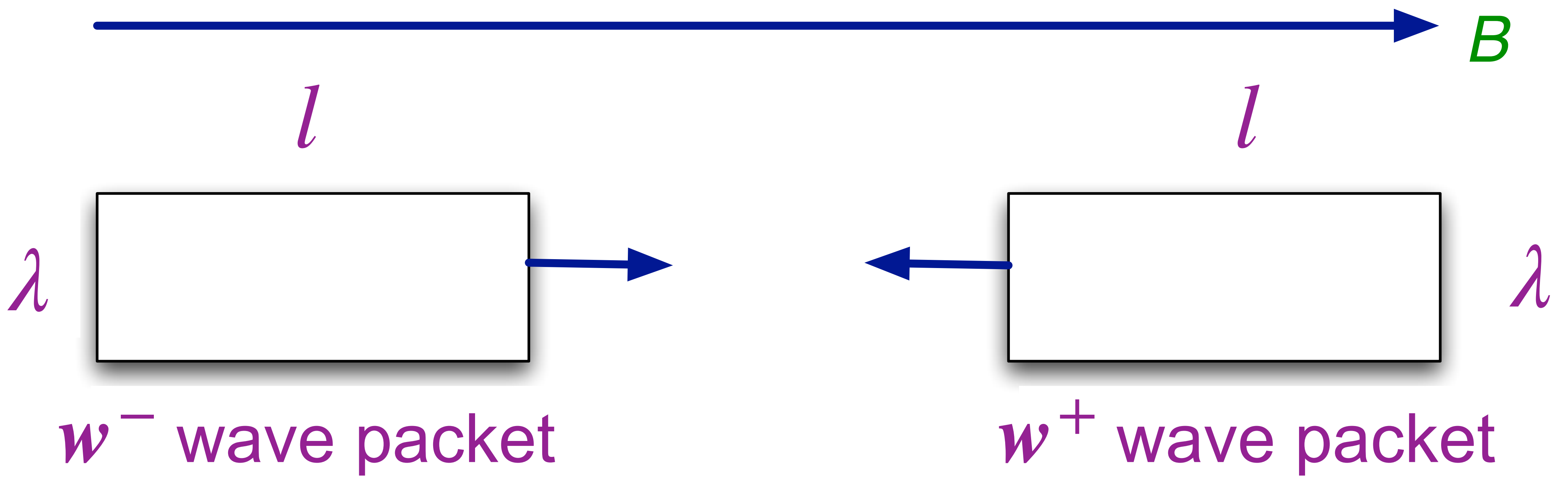

Let's see if we can derive the inertial-range power spectrum for weak, incompressible MHD turbulence using the same types of arguments that we reviewed yesterday when discussing Kolmogorov (1941) famous $k^{-5/3}$ scaling for hydrodynamic turbulence.

(Goldreich & Sridhar 1997, Ng & Bhattacharjee 1997)

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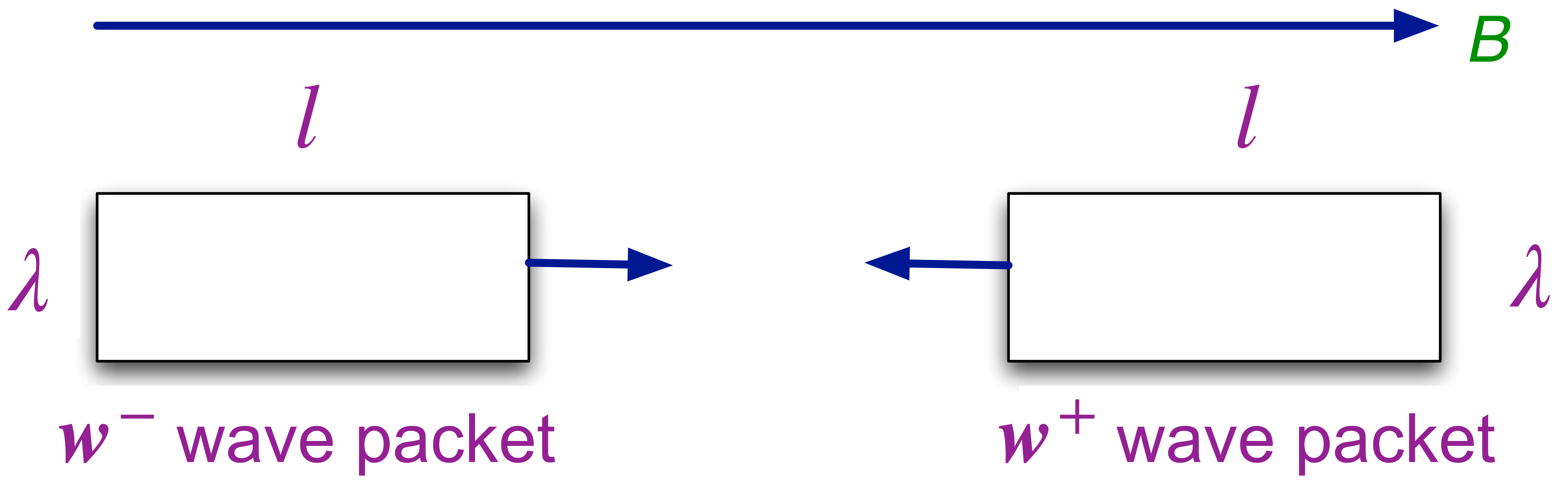

- w_λ = r.m.s. increment in $\boldsymbol{w}^{\pm}$ across a distance λ in plane $\perp$ to $\boldsymbol{B}$ $\sim$ velocity fluctuation of wave packet

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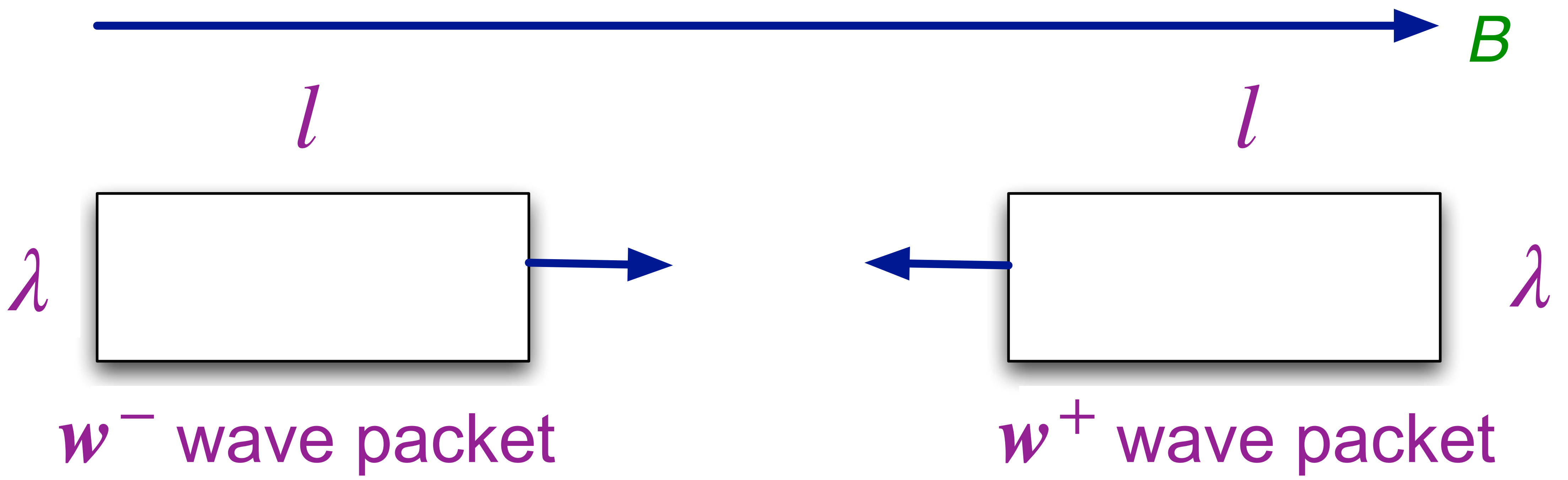

- w_λ = r.m.s. increment in $\boldsymbol{w}^{\pm}$ across a distance λ in plane $\perp$ to $\boldsymbol{B}$ $\sim$ velocity fluctuation of wave packet
- Contribution of wave packets at $\perp$ scale λ to $\boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$ is $\sim w_\lambda^2/\lambda$. (Note: I am considering Alfvén wave packets, for which $\boldsymbol{w}^{\pm}$ is approximately $\perp$ to $\boldsymbol{B}$. This is why $\nabla \to 1/\lambda$ rather than $1/l$.)

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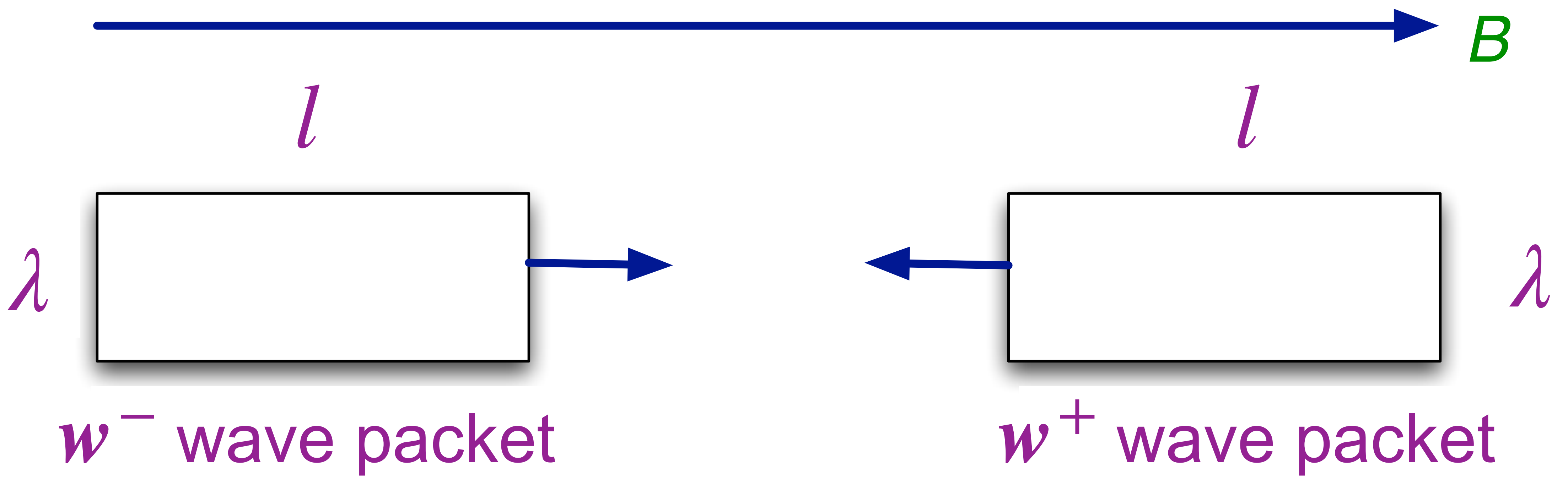

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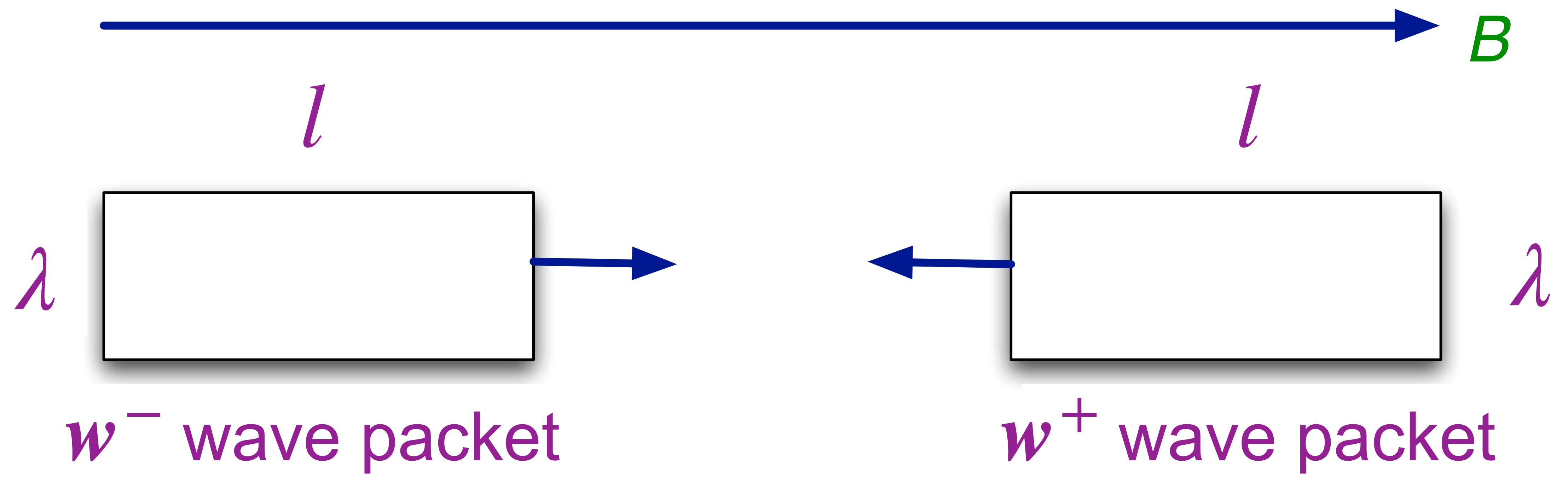

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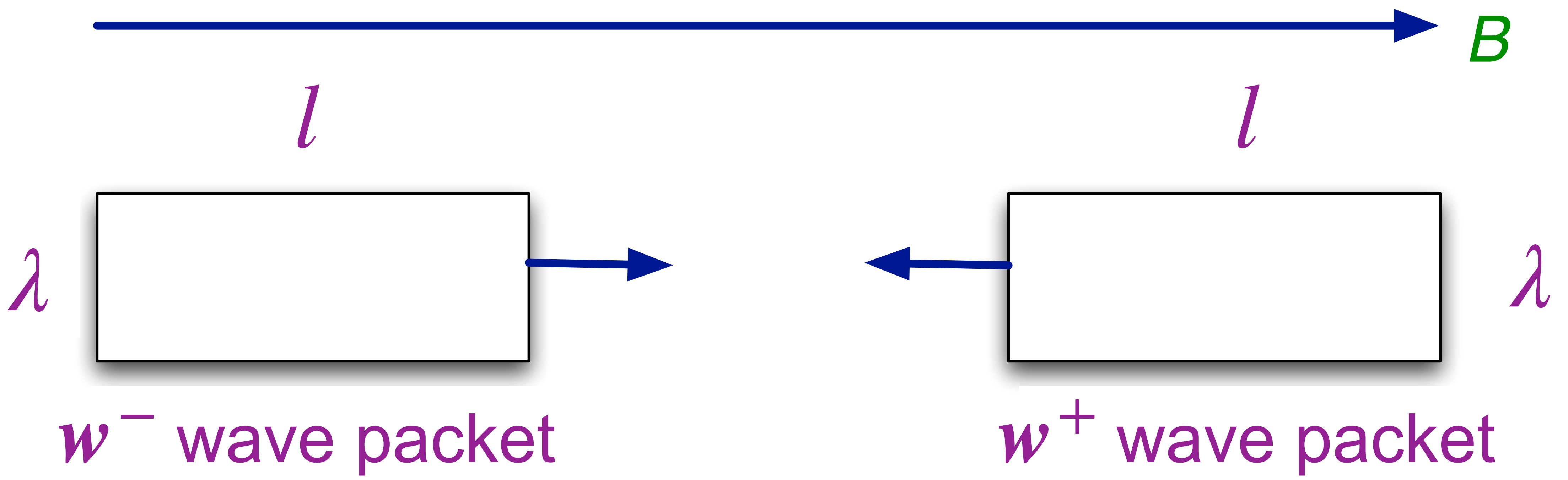

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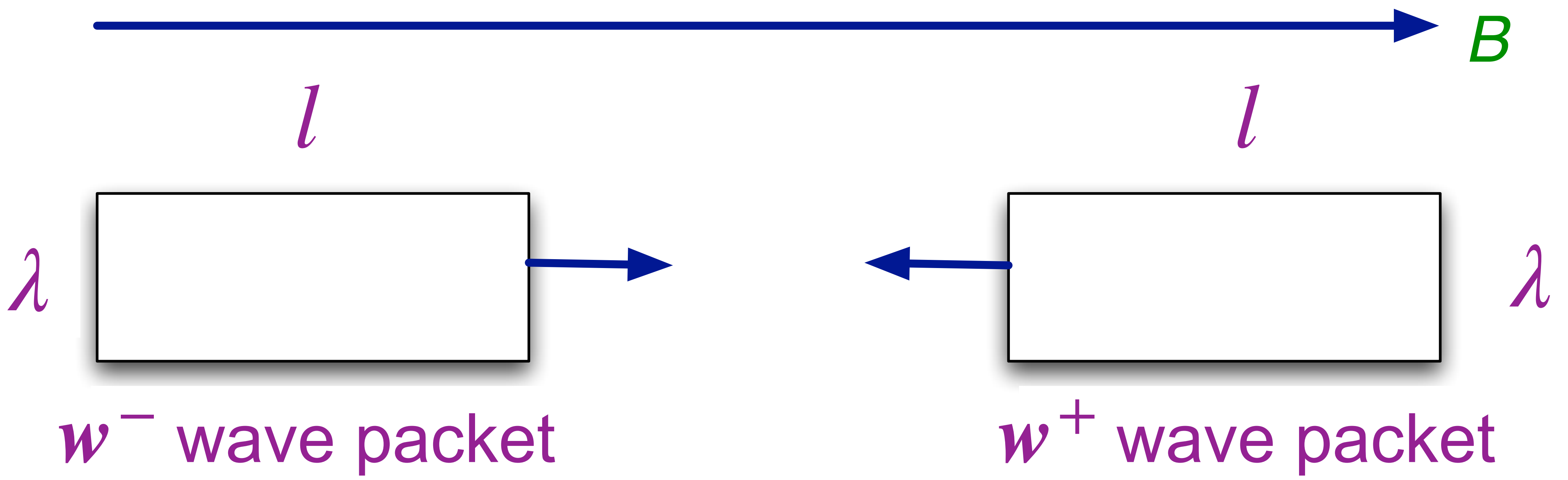

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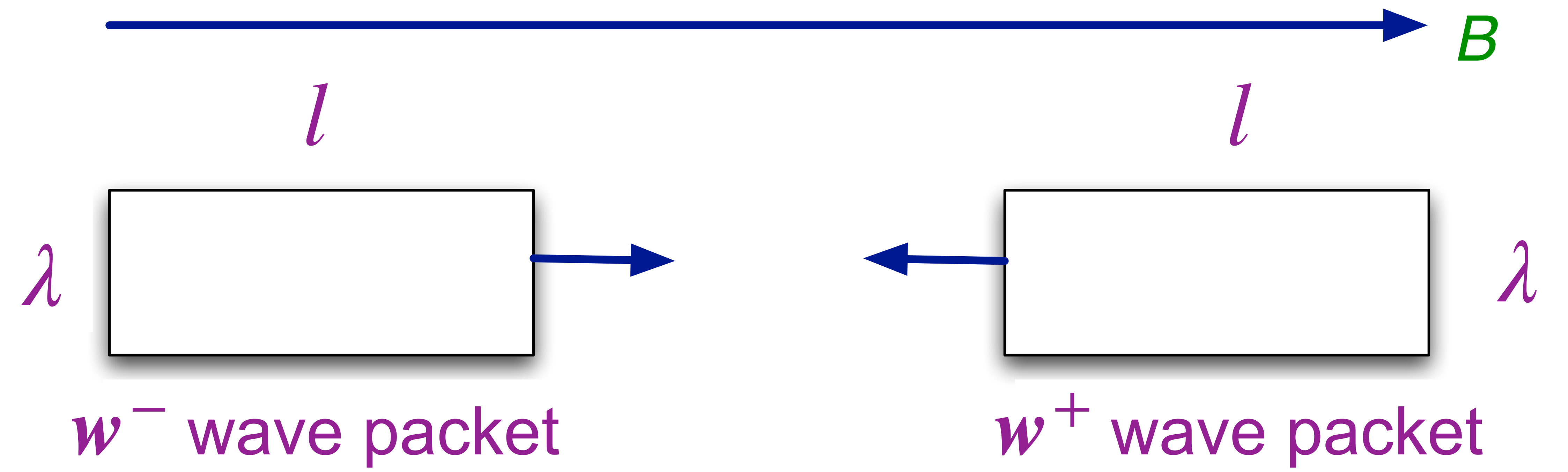

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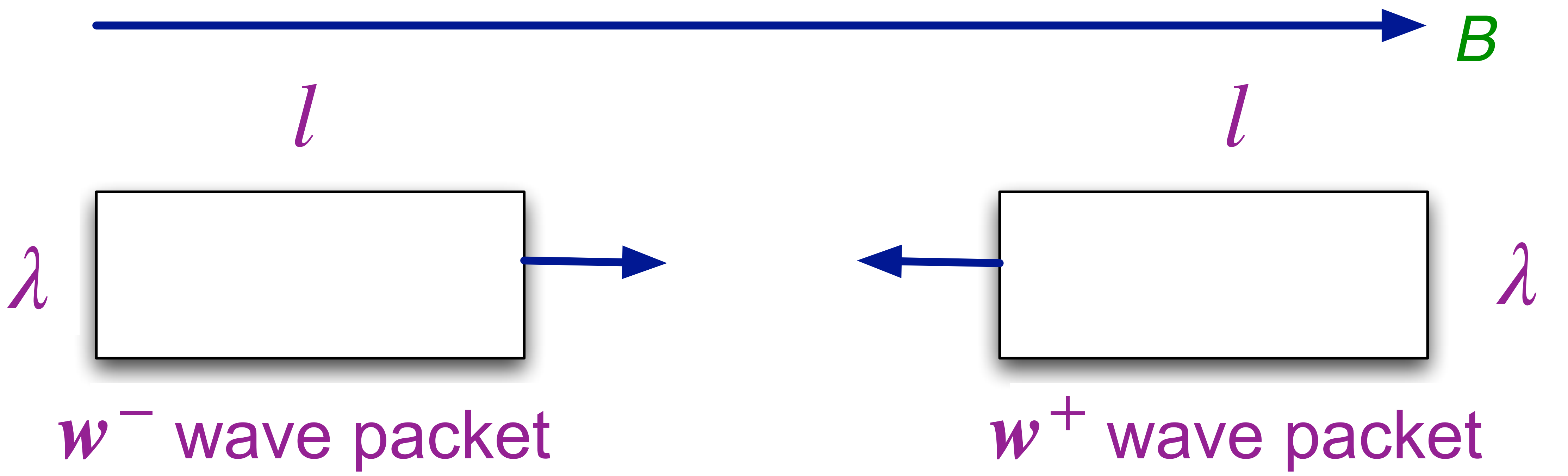

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(Goldreich & Sridhar 1997, Ng & Bhattacharjee 1997)

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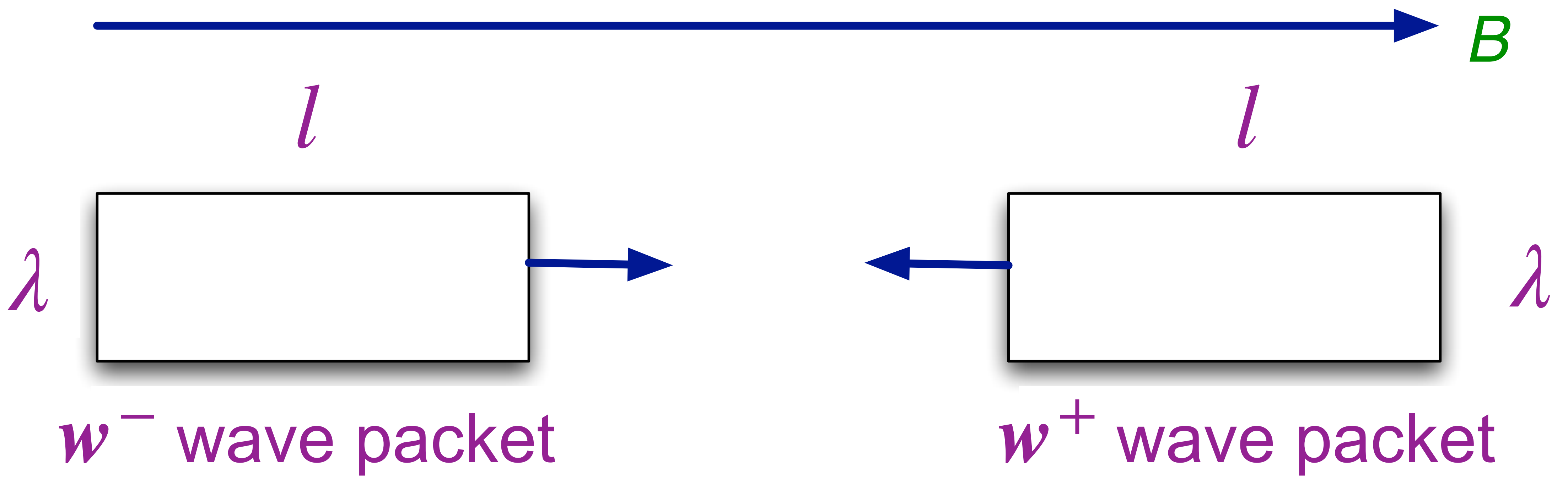

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- $\longrightarrow$ it takes $N \sim \chi^{-2}$ collisions before a wave packet's energy cascades to smaller scales, and the energy cascade time is $\tau_{\rm c} \sim \chi^{-2}\Delta t = \frac{\lambda^2 v_{\rm A}^2}{l^2 w_\lambda^2} \times \frac{l}{v_{\rm A}} = \frac{\lambda^2 v_{\rm A}}{l w_\lambda^2}$

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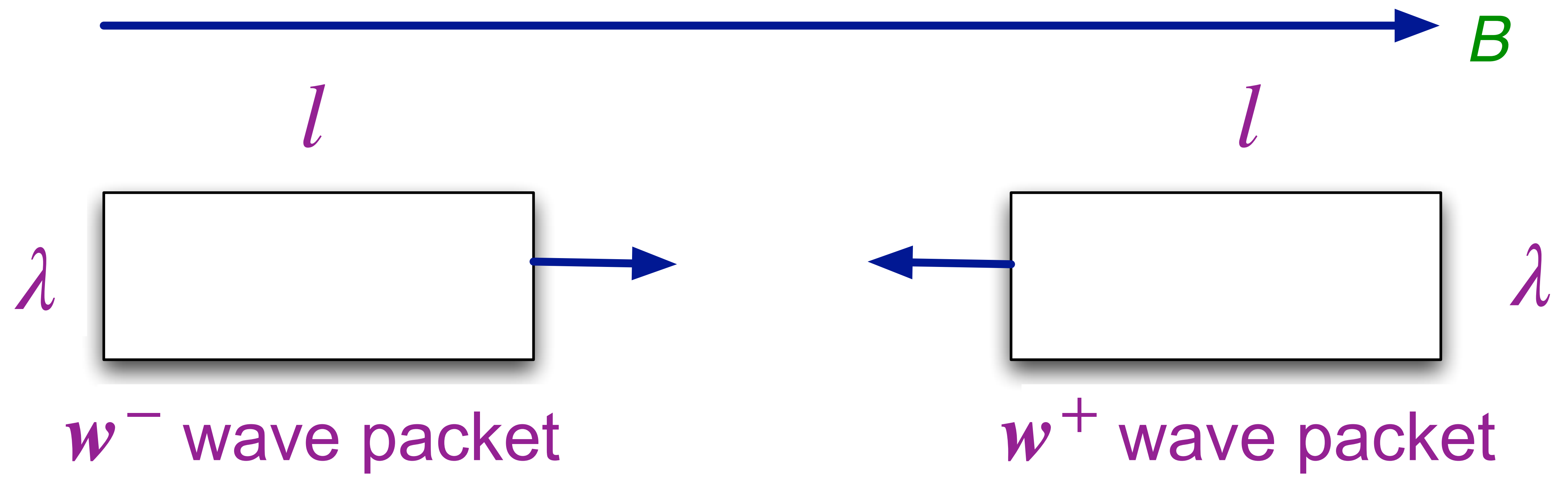

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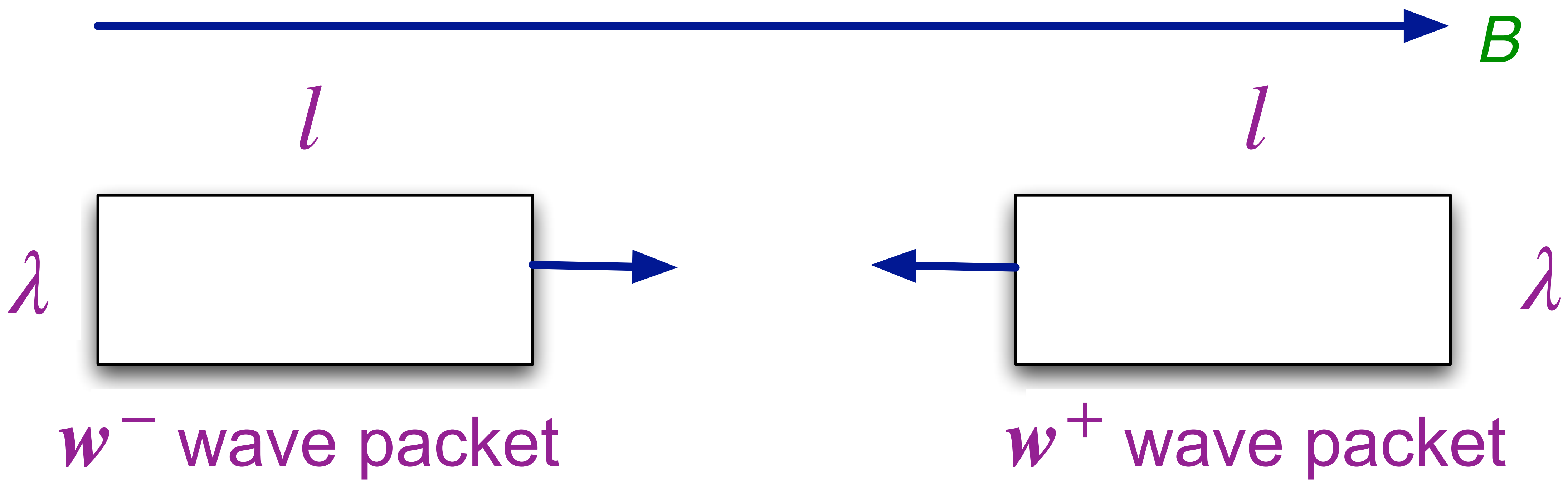

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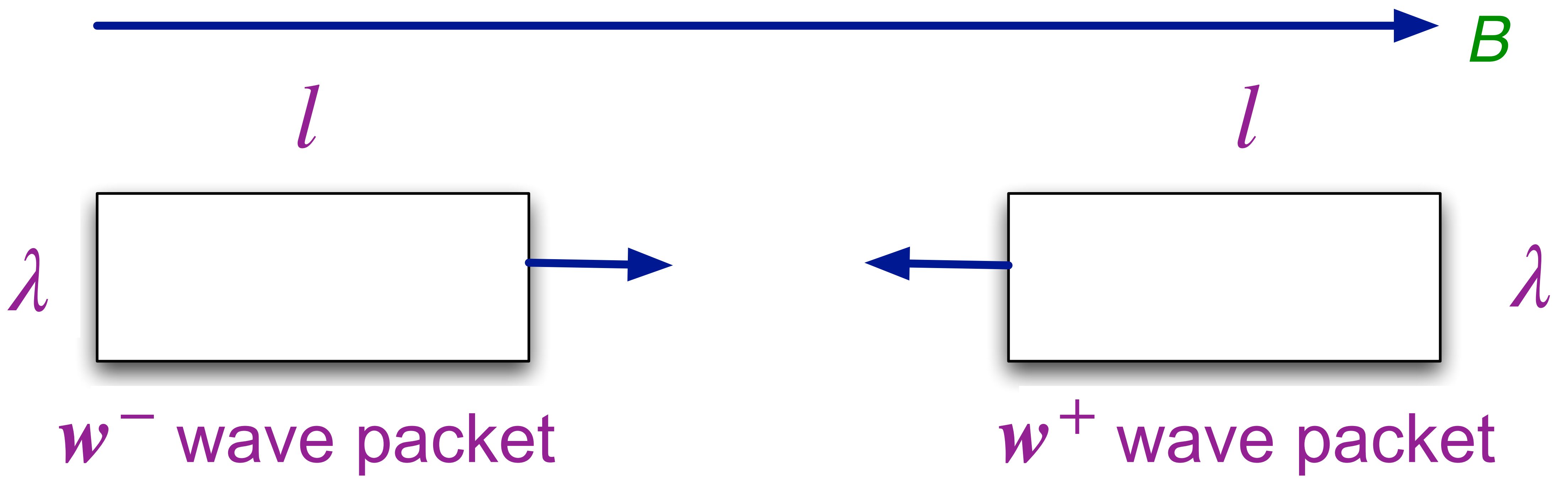
- As in hydro turbulence, within the inertial range the cascade power ϵ is independent of λ, where $\epsilon \sim \frac{w_\lambda^2}{\tau_{\rm c}} \sim \frac{w_\lambda^4 l}{\lambda^2 v_{\rm A}}$. This means that $w_\lambda \propto \lambda^{1/2}$, because l is constant.

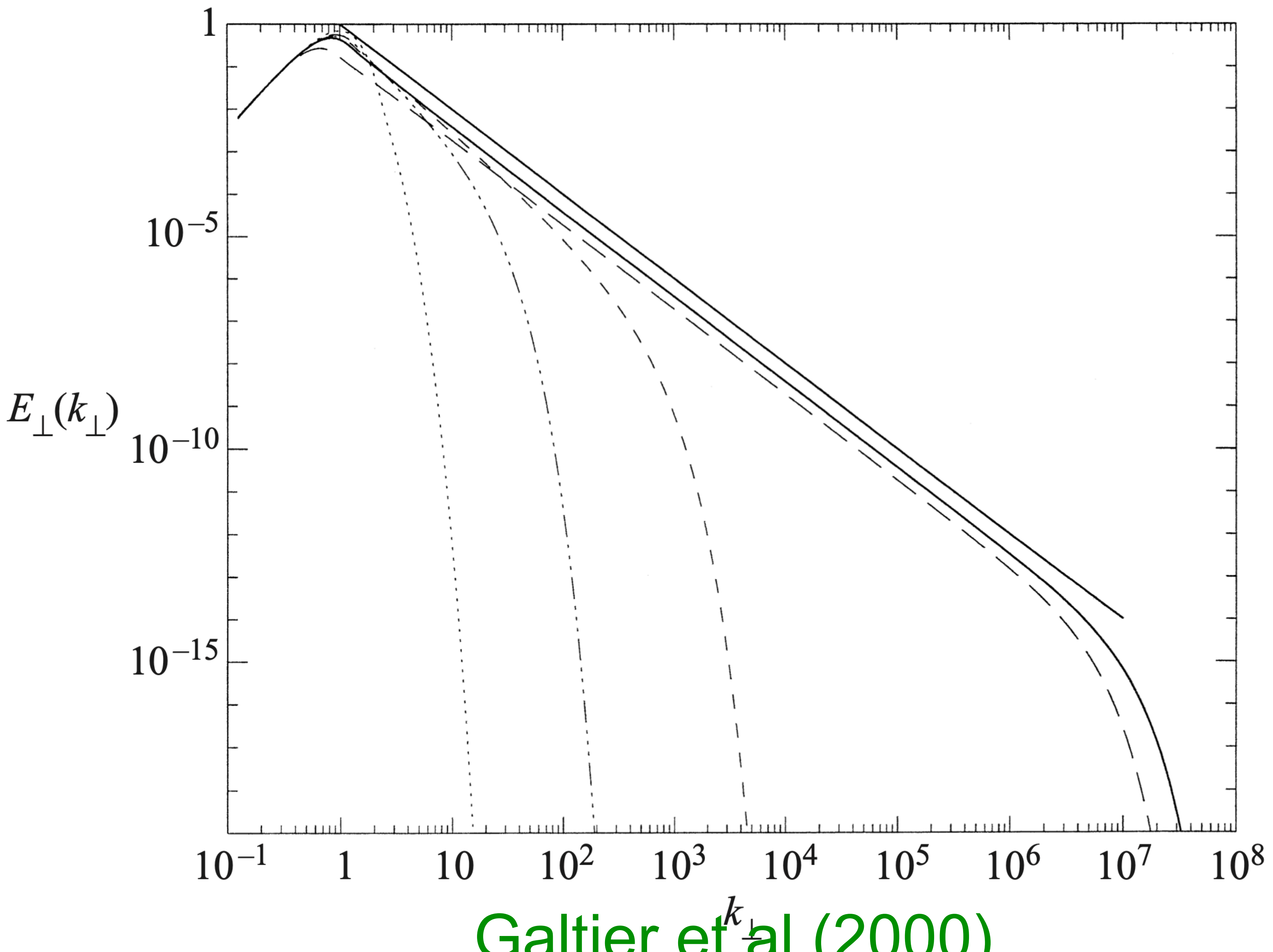
(Goldreich & Sridhar 1997, Ng & Bhattacharjee 1997)

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\mathrm{A}} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla\Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$$



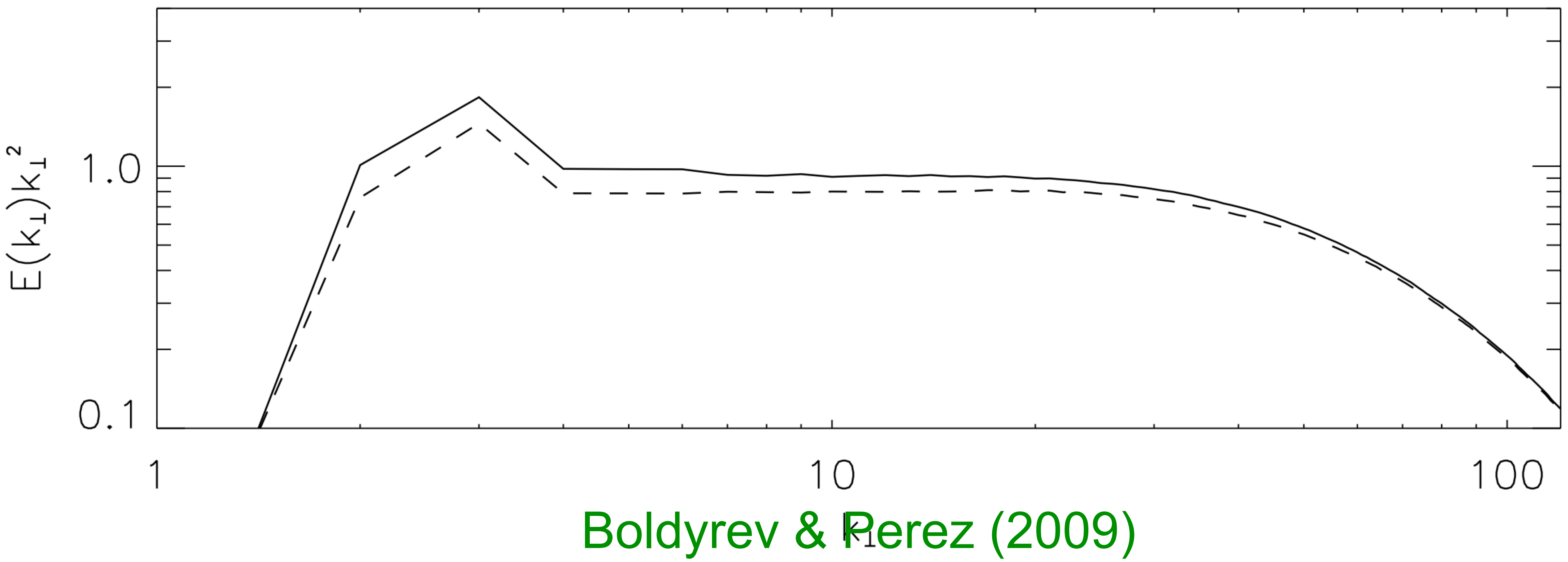
- $\longrightarrow$ it takes $N \sim \chi^{-2}$ collisions before a wave packet's energy cascades to smaller scales, and the energy cascade time is $\tau_{\mathrm{c}} \sim \chi^{-2}\Delta t = \dfrac{\lambda^2 v_{\mathrm{A}}^2}{l^2 w_\lambda^2} \times \dfrac{l}{v_{\mathrm{A}}} = \dfrac{\lambda^2 v_{\mathrm{A}}}{l w_\lambda^2}$
- As in hydro turbulence, within the inertial range the cascade power ϵ is independent of λ, where $\epsilon \sim \dfrac{w_\lambda^2}{\tau_{\mathrm{c}}} \sim \dfrac{w_\lambda^4 l}{\lambda^2 v_{\mathrm{A}}}$. This means that $w_\lambda \propto \lambda^{1/2}$, because l is constant.
- $k_\perp E(k_\perp) \equiv \left(w_\lambda^2\right)_{\lambda=1/k_\perp} \longrightarrow k_\perp E(k_\perp) \propto k_\perp^{-1} \longrightarrow \boxed{E(k_\perp) \propto k_\perp^{-2}}$

Numerical Examples of $k_\perp^{-2}$ Inertial-Range Power Spectra in Weak Incompressible MHD Turbulence



Galtier et al (2000)

(based on weak turbulence theory)



Boldyrev & Perez (2009)

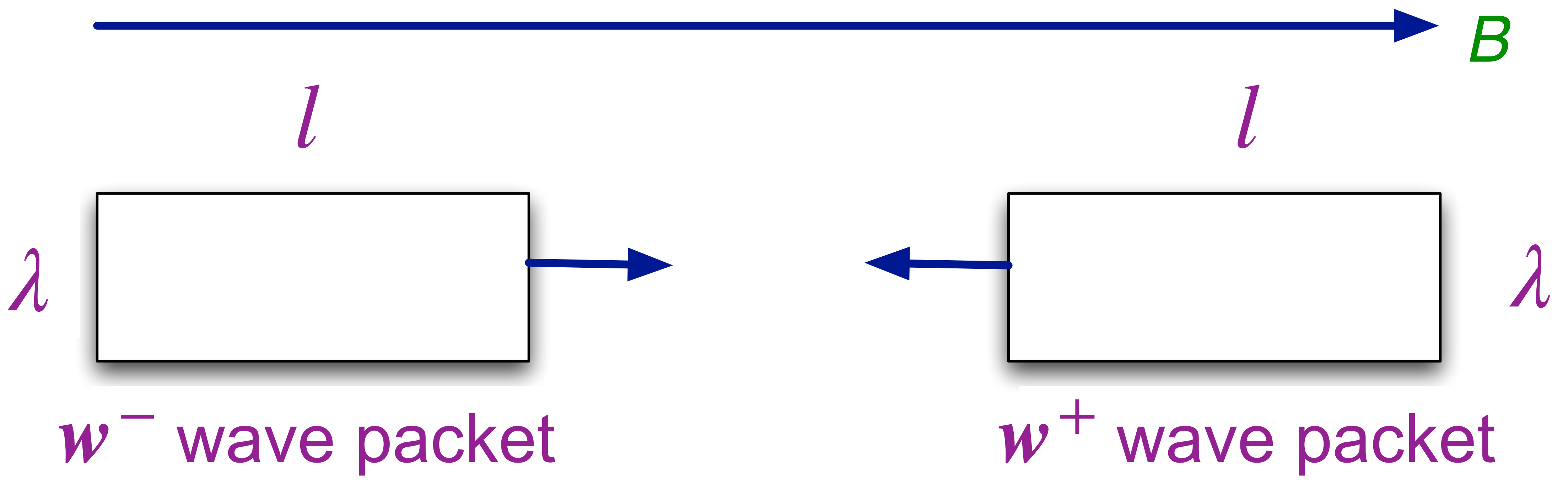
(from direct numerical simulations of the incompressible MHD equations)

Outline

1. Quick review of magnetohydrodynamics (MHD)
2. Elsässer form of the incompressible MHD equations
3. Linear waves, weak turbulence, and strong turbulence
4. Weak incompressible MHD turbulence and the anisotropic energy cascade
5. Strong incompressible MHD turbulence and critical balance
6. Extras: compressible turbulence, inverse cascade of magnetic helicity helicity barrier, cosmic-ray scattering by MHD turbulence

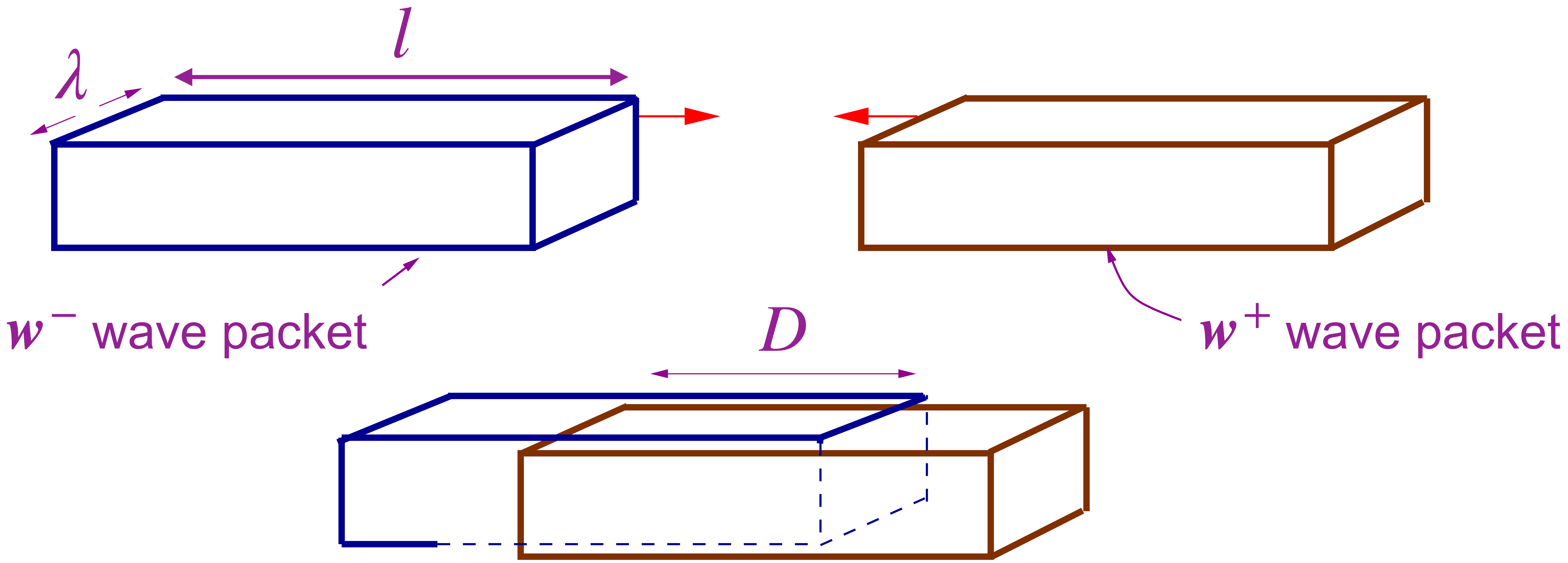
(Goldreich & Sridhar 1995, 1997)

$$\frac{\partial}{\partial t}\boldsymbol{w}^{\pm} \mp \boldsymbol{v}_{\rm A} \cdot \nabla \boldsymbol{w}^{\pm} = -\nabla\Pi - \boldsymbol{w}^{\mp} \cdot \nabla \boldsymbol{w}^{\pm}$$



- From before, the fractional change of $\boldsymbol{w}^{\pm}$ in each wave packet during 1 collision is $\chi \sim \dfrac{w_\lambda l}{\lambda v_{\rm A}} \sim \dfrac{\tau_{\rm linear}}{\tau_{\rm nonlinear}}$, where $\tau_{\rm linear} = l/v_{\rm A}$ is like the linear Alfvén wave period, and $\tau_{\rm nonlinear}^{-1} = w_\lambda/\lambda$ is the shearing rate of eddies at $\perp$ scale λ
- $w_\lambda \propto \lambda^{1/2}$ and $l \propto \lambda^0 \longrightarrow \chi \propto \lambda^{-1/2}$. As λ decreases, eventually χ grows to a value ~ 1, and the turbulence becomes strong

What Happens If l Is Initially So Large That $\chi \gg 1$



- After colliding wave packets have inter-penetrated by a distance D satisfying $\frac{D}{v_A} \times \frac{w_\lambda}{\lambda} \sim 1$, the leading edge of each wave packet will have been substantially sheared/altered relative to the trailing edge. Nonlinear interactions therefore reduce l until $l \lesssim D$ and decrease χ until $\chi = \frac{l}{v_A} \times \frac{w_\lambda}{\lambda} \lesssim 1$.

- In weak turbulence, $\chi \ll 1$ but χ grows to ~ 1 as λ decreases. If $\chi \gg 1$, then nonlinear interactions reduce χ to ~ 1. Incompressible MHD turbulence thus gravitates towards a state of critical balance in which $\chi \sim 1$ (Goldreich & Sridhar 1995). If the turbulence starts at $\chi \sim 1$ at some scale λ, it maintains $\chi \sim 1$ at smaller scales.

The Kolmogorov-Like Power Spectrum of Critically Balanced MHD Turbulence

(Goldreich & Sridhar 1995)

- In strong incompressible MHD turbulence, $\chi \sim 1$, and the energy cascade time is $\tau_{\rm c} \sim \lambda / w_\lambda$, just like the hydro-turbulence cascade time scale in yesterday's talk was $\sim \lambda / u_\lambda$.

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- $\chi = \dfrac{w_\lambda l}{\lambda v_A} \sim 1 \quad \longrightarrow \quad l \propto \lambda / w_\lambda \propto \lambda^{2/3}$. This implies that $\dfrac{l}{\lambda} \propto \lambda^{-1/3}$ — eddies become more anisotropic as λ decreases. Defining $k_\parallel = 1/l$ and $k_\perp = 1/\lambda$, we get $k_\parallel \propto k_\perp^{2/3}$

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- $k_\perp E(k_\perp) \equiv \left(w_\lambda^2 \right)_{\lambda = 1/k_\perp} \longrightarrow E(k_\perp) \propto k_\perp^{-5/3}$

Numerical Simulations of Strong Incompressible MHD Turbulence

(Cho & Lazarian 2000)

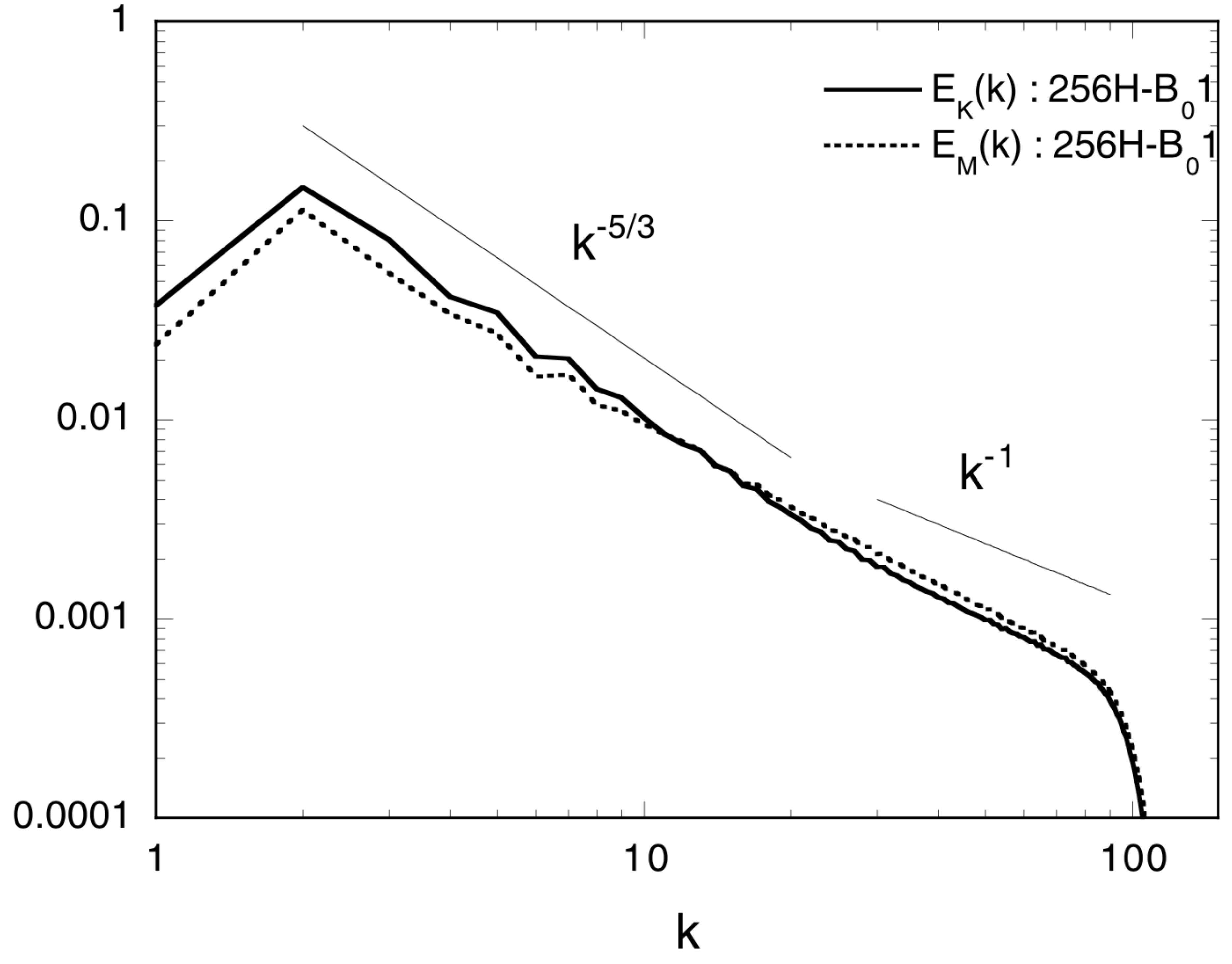


FIG. 2.—256*H*-B_0 1. Kinetic energy spectrum ($E_K(k)$) and magnetic energy spectrum ($E_M(k)$). For $2 \leq k \leq 20$, spectra are compatible with $k^{-5/3}$. A $1/k$ bottleneck effect is observed before the dissipation cutoff $k_d \sim 90$.

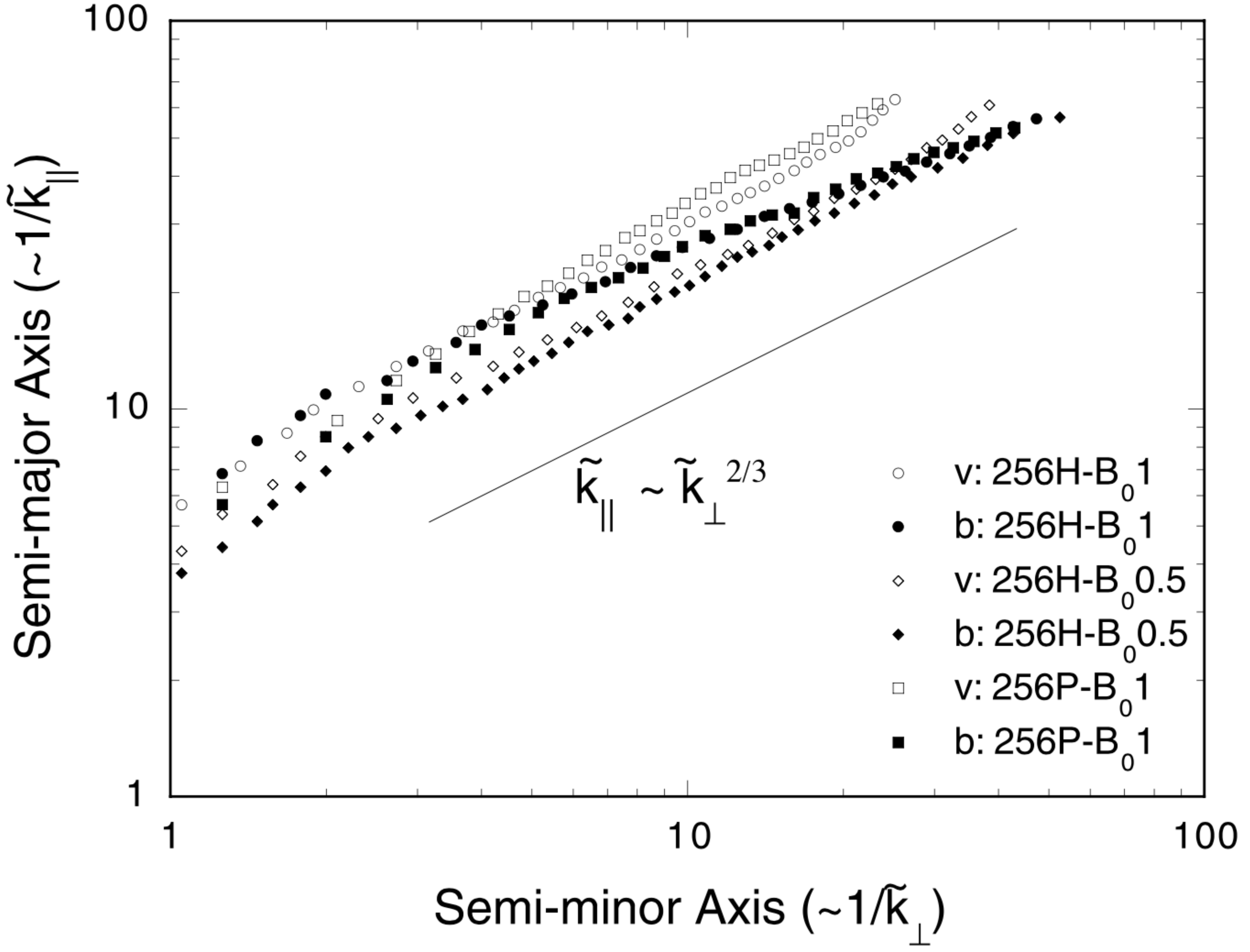
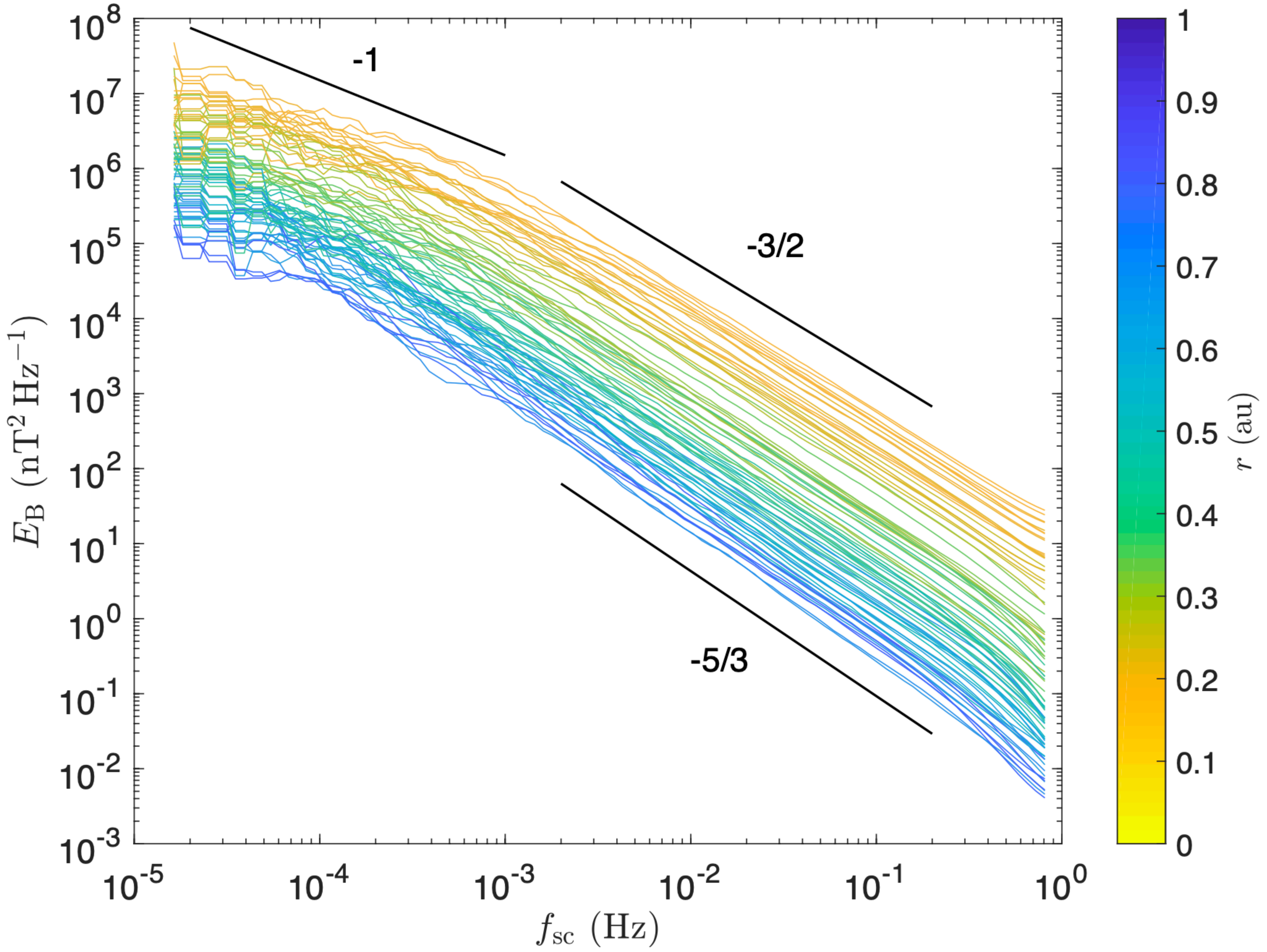


FIG. 9.—*R*-intercept (semiminor axis; $\sim 1/\tilde{k}_\perp$) vs. *z*-intercept (semimajor axis; $\sim 1/\tilde{k}_\parallel$) from Fig. 8. In 256*H*-B_0 0.5, both velocity and magnetic fields follow the relation $\tilde{k}_\parallel \sim \tilde{k}_\perp^{2/3}$. In 256*H*-$B_0$ 1 and 256*P*-B_0 1, velocity fields follow the same scaling relation. However, magnetic fields scale slightly differently.

Solar Wind Turbulence

(Chen et al 2020 — Parker Solar Probe measurements)



Other Topics

1. Intermittency
2. Compressibility
3. Dynamic Alignment
4. Imbalance
5. Kintetic Alfvén wave turbulence
6. Helicity barrier
7. Spherically polarized Alfvén waves and switchbacks
8. Cosmic-ray scattering

Conclusion

- In incompressible MHD turbulence, nonlinear interactions occur only between counter-propagating wave packets.

- In weak incompressible MHD turbulence: (1) there is no parallel cascade; (2) $E(k_\perp) \propto k_\perp^{-2}$, and (3) at sufficiently small scales the critical balance parameter χ increases to 1, and the turbulence becomes strong.

- In strong incompressible MHD turbulence: (1) $\chi \sim 1$ at all scales and the turbulence remains strong throughout the inertial range; (2) $E(k_\perp) \propto k_\perp^{-5/3}$; and (3) $l \propto \lambda^{2/3}$, implying that the eddies or wave packets become increasingly anisotropic as you go to smaller λ.