Introduction to Hydrodynamic Turbulence

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Goals

- To review a few things from earlier this week, as doing so can help key ideas to sink in.
- To 'de-mystify' the subject of turbulence for you.
- To show you a few classic (and broadly useful) results, but also to explain carefully how you can recover those results for yourself.
- This means:
  - Wherever possible, I'm going to show you all the steps.
  - Much of this talk will be dry/mathematical, and there are many cool ideas and results that I will not have time to share with you (sorry).
  - But for many of you, this level of talk is not available elsewhere. Conference talks are way too advanced for students trying to learn about turbulence, and classes often don't cover this material. So I think you will find this useful, and worth your careful attention.

Outline

1. Intro: what is turbulence?
2. Review: continuity equation, Navier-Stokes equation
3. Review: Fourier transforms
4. Energy cascade from large scales to small scales in 3D hydro
5. Inverse energy cascade from small scales to large scales in 2D hydro
6. Extras: turbulent transport, turbulent heating, passive-scalar diffusion

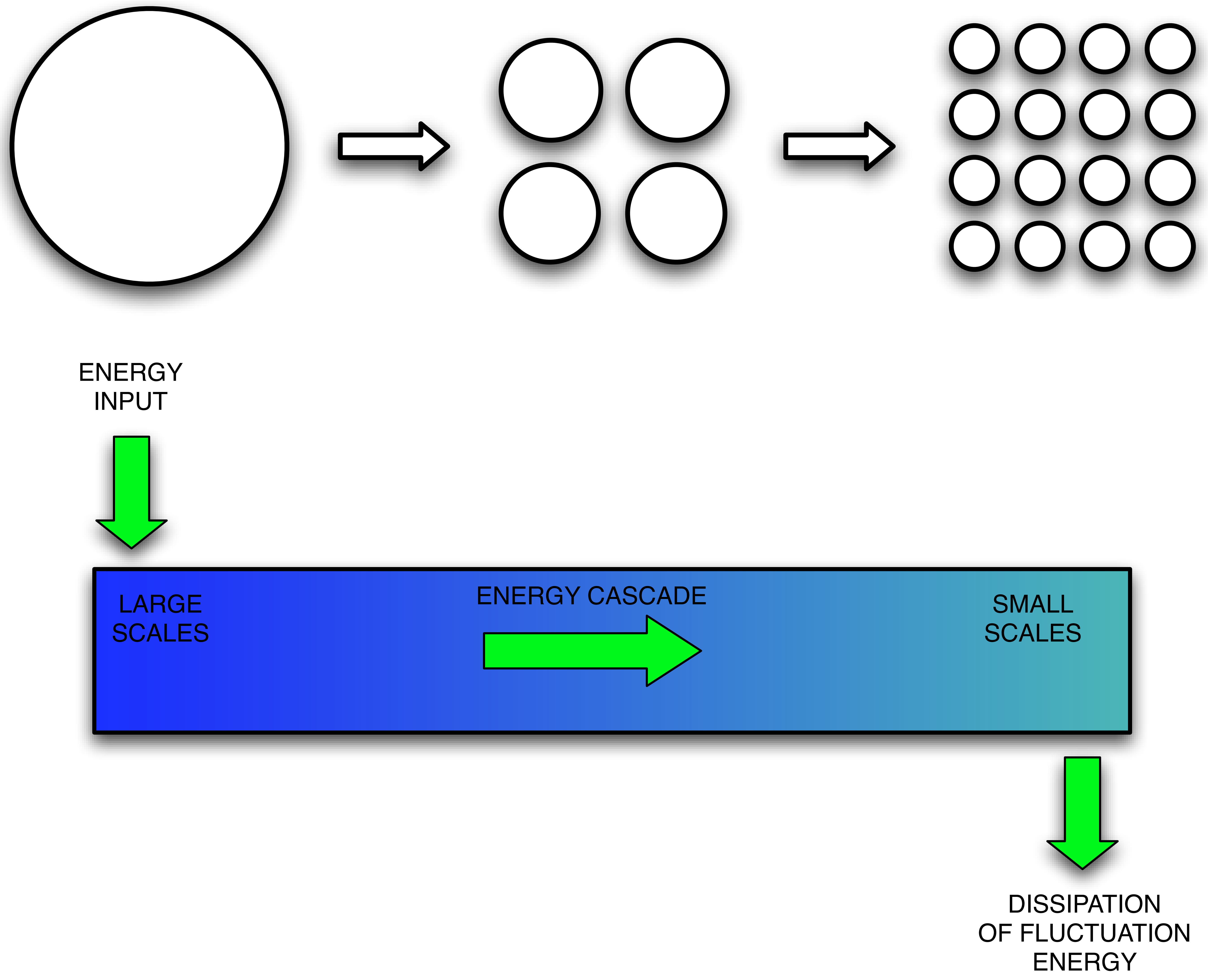
I probably won't be able to get through all these points, but I'll make the slides available. Please interrupt to ask questions!

Turbulence: What Is It?

Turbulence consists of disordered, interacting fluctuations (e.g., in the flow velocity) that span a broad range of scales in space and time.

Hydrodynamic Turbulence

Canonical picture: larger eddies break up into smaller eddies



Turbulence: Where Do You Find It?

- Atmosphere (driven by convective currents or flow over obstacles)
- Ocean (driven by wind at surface)
- Sun (driven by convection), solar corona, solar wind
- Accretion flows, star-forming molecular clouds, other phases of the interstellar medium, the intracluster plasma of galaxy clusters
- Almost everywhere — so a good thing for any astrophysicist to understand.

Turbulence: What Does It Do?

- Transports energy, e.g., from the Sun's core to the Sun's surface. Eddy motions cause the thermal energy to undergo a random walk in space, leading to the outward diffusion of energy.

- Transports angular momentum, e.g., outward within an accretion disk.

- Turbulent mixing (passive scalar diffusion) — e.g., cream into coffee, or the diffusion of different elements within the convection zone of a star.

- Turbulent heating — e.g., heating of the solar corona, solar wind, or intracluster plasma in galaxy clusters.

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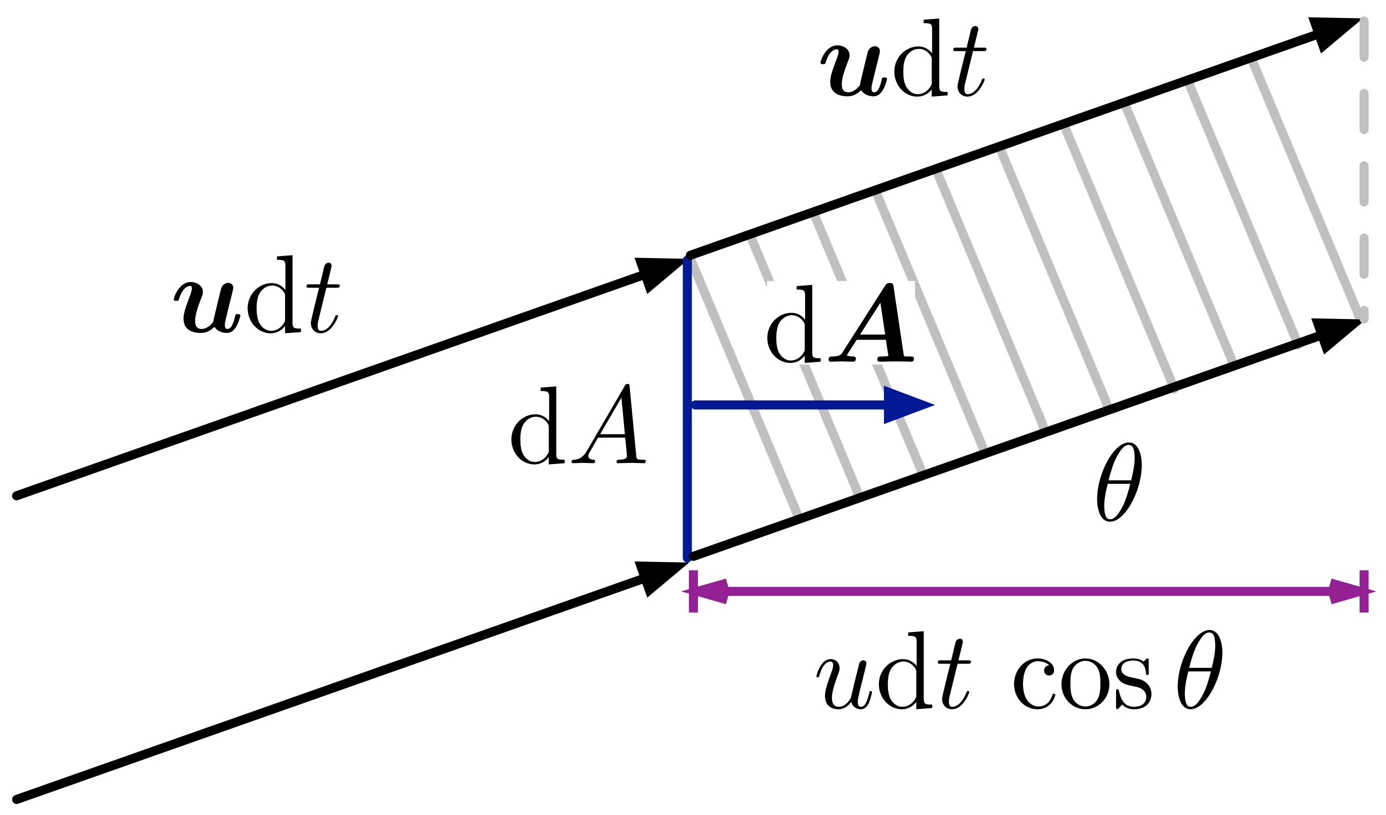
Mass Flux

ρ = mass density $\qquad \boldsymbol{u}$ = fluid velocity



(vectors in bold italic font)

$$\text{cross-hatched volume} = \mathrm{d}A\, u\mathrm{d}t\, \cos\theta = \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{A}\, \mathrm{d}t$$

During time $\mathrm{d}t$, fluid flowing through infinitesimal surface element $\mathrm{d}\boldsymbol{A}$ fills the cross-hatched volume and has mass $\mathrm{d}M = \rho\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A}\,\mathrm{d}t$.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \rho\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A} = \text{ rate at which mass flows through } \mathrm{d}\boldsymbol{A} \text{ per unit time}$$

$\boldsymbol{F}_{\mathrm{mass}} \equiv \text{mass flux} = \rho\boldsymbol{u} =$ rate at which mass flows through a (normal) surface per unit area per unit time.

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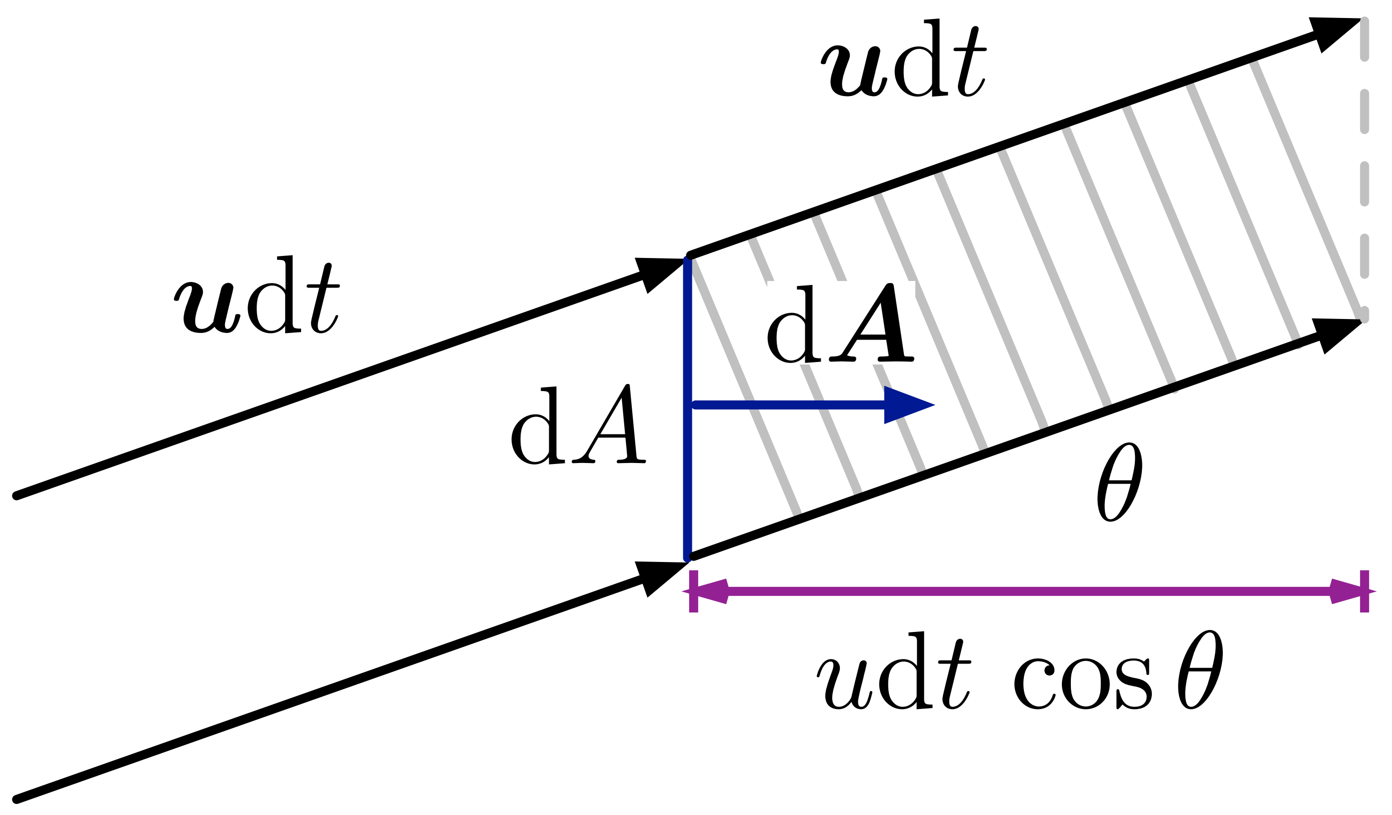

cross-hatched volume $= \mathrm{d}A\, u\mathrm{d}t\,\cos\theta = \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A}\,\mathrm{d}t$

General idea:

flux = density × velocity

We'll see this again tomorrow for the charge flux.

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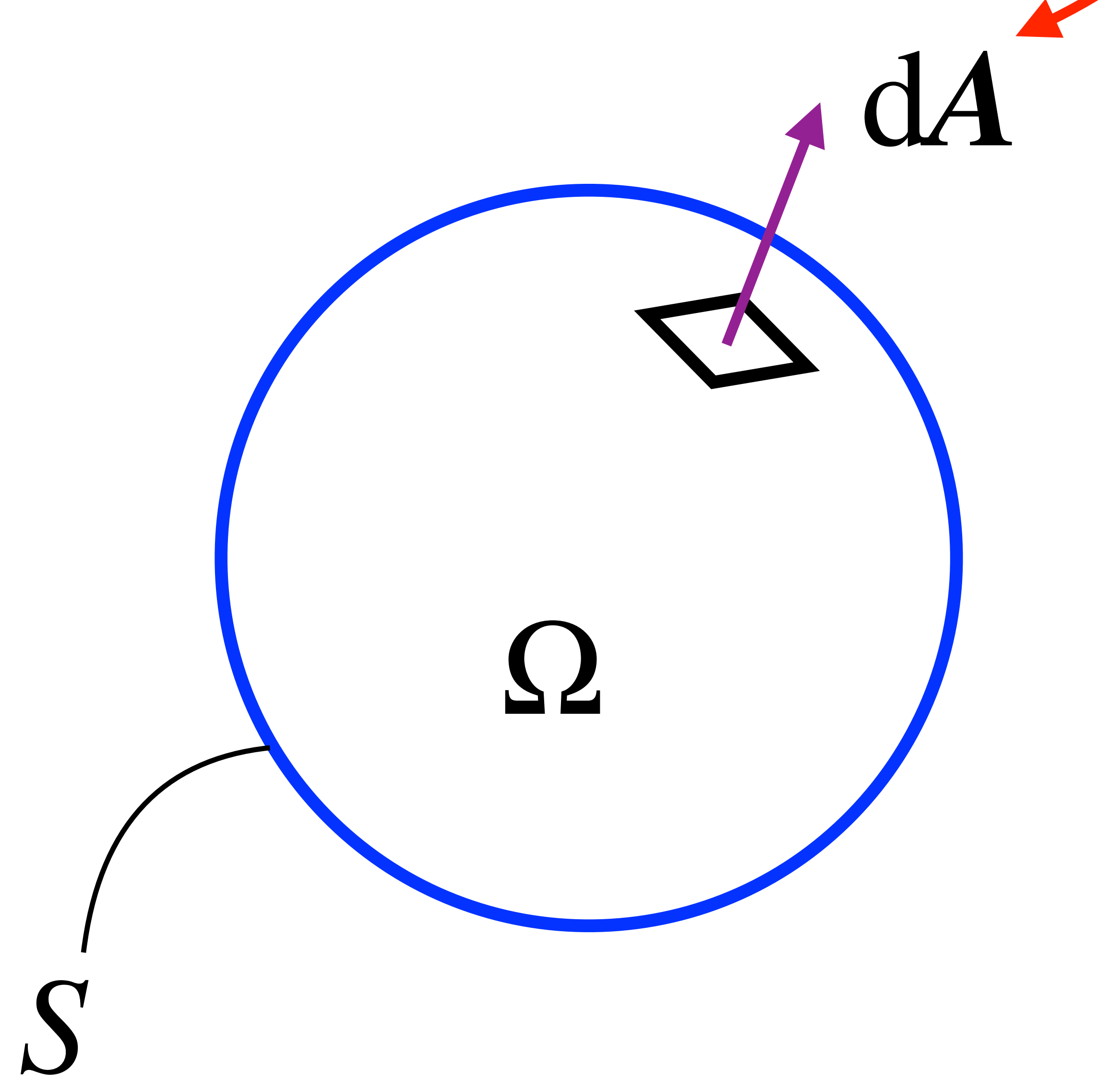
Conservation of Mass

(vectors in bold italic font)

Consider an arbitrary fixed volume Ω with boundary S within a fluid with density $\rho(\boldsymbol{x}, t)$ and flow velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The mass within Ω is $M = \int_\Omega \rho \, \mathrm{d}^3 x$.



$\mathrm{d}M/\mathrm{d}t$ is just the rate at which mass flows in through the boundary of Ω

$$\longrightarrow \int_\Omega \frac{\partial \rho}{\partial t} \mathrm{d}^3 x = -\oint_S \rho \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} = -\int_\Omega \nabla \cdot (\rho \boldsymbol{u}) \, \mathrm{d}^3 x$$

(by Gauss's theorem)

As Ω is arbitrary, $\dfrac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u})$ everywhere

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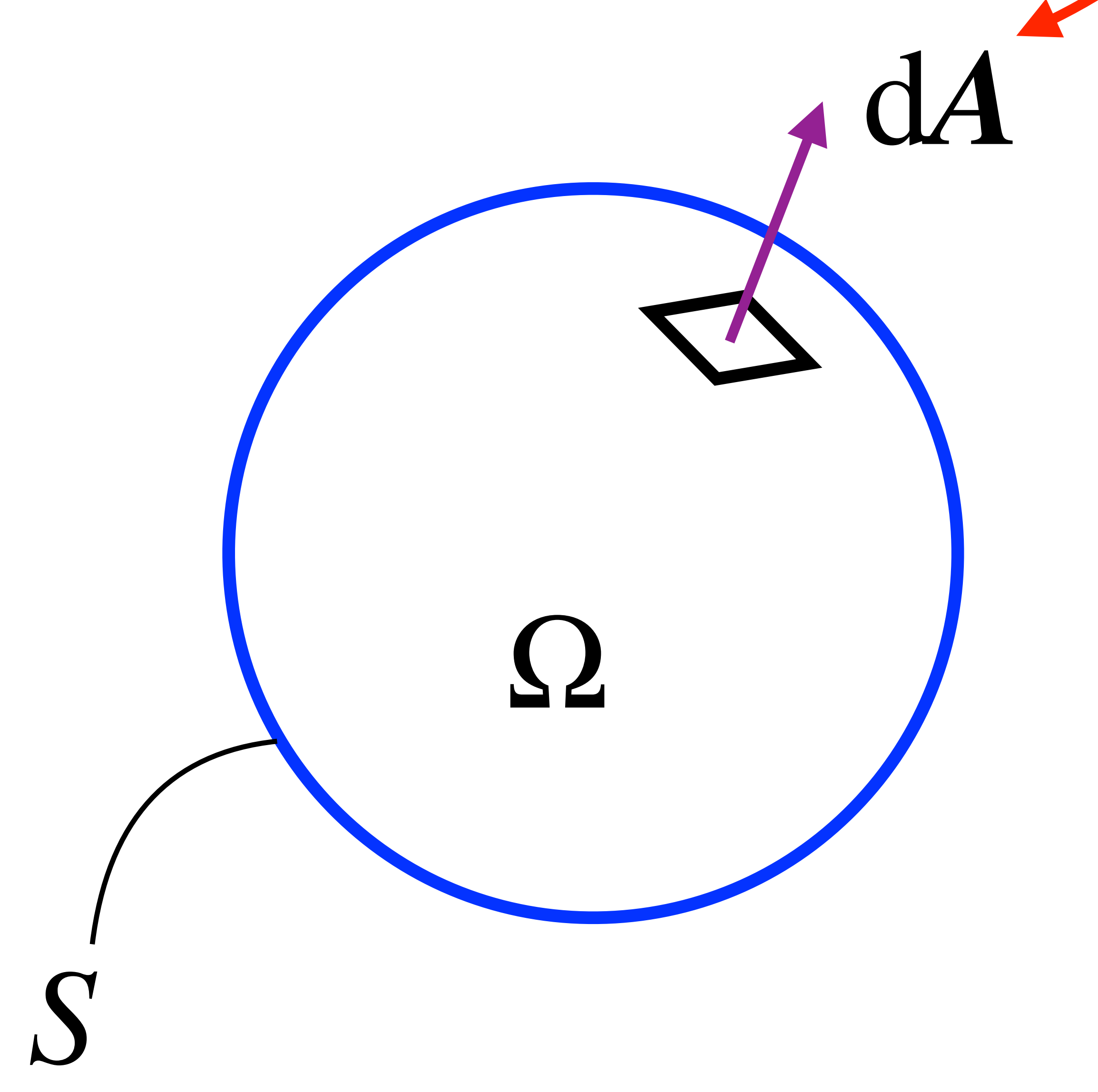

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As Ω is arbitrary, $$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u})}$$ everywhere — 'continuity equation'

Newton's Second Law: $\boldsymbol{a} = \boldsymbol{F}/m$

Suppose a fluid has velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. Is $\boldsymbol{a} = \dfrac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x}, t)$?

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Consider a fluid element with position $\boldsymbol{x}(t)$ and velocity $\boldsymbol{u}(\boldsymbol{x}(t), t)$. Then

$$\boldsymbol{u}(\boldsymbol{x}(t), t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)$$

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$$\boldsymbol{a} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{x}(t), t) = \left(\frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\nabla\right)\boldsymbol{u}(\boldsymbol{x}(t), t) = \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}(\boldsymbol{x}(t), t)$$

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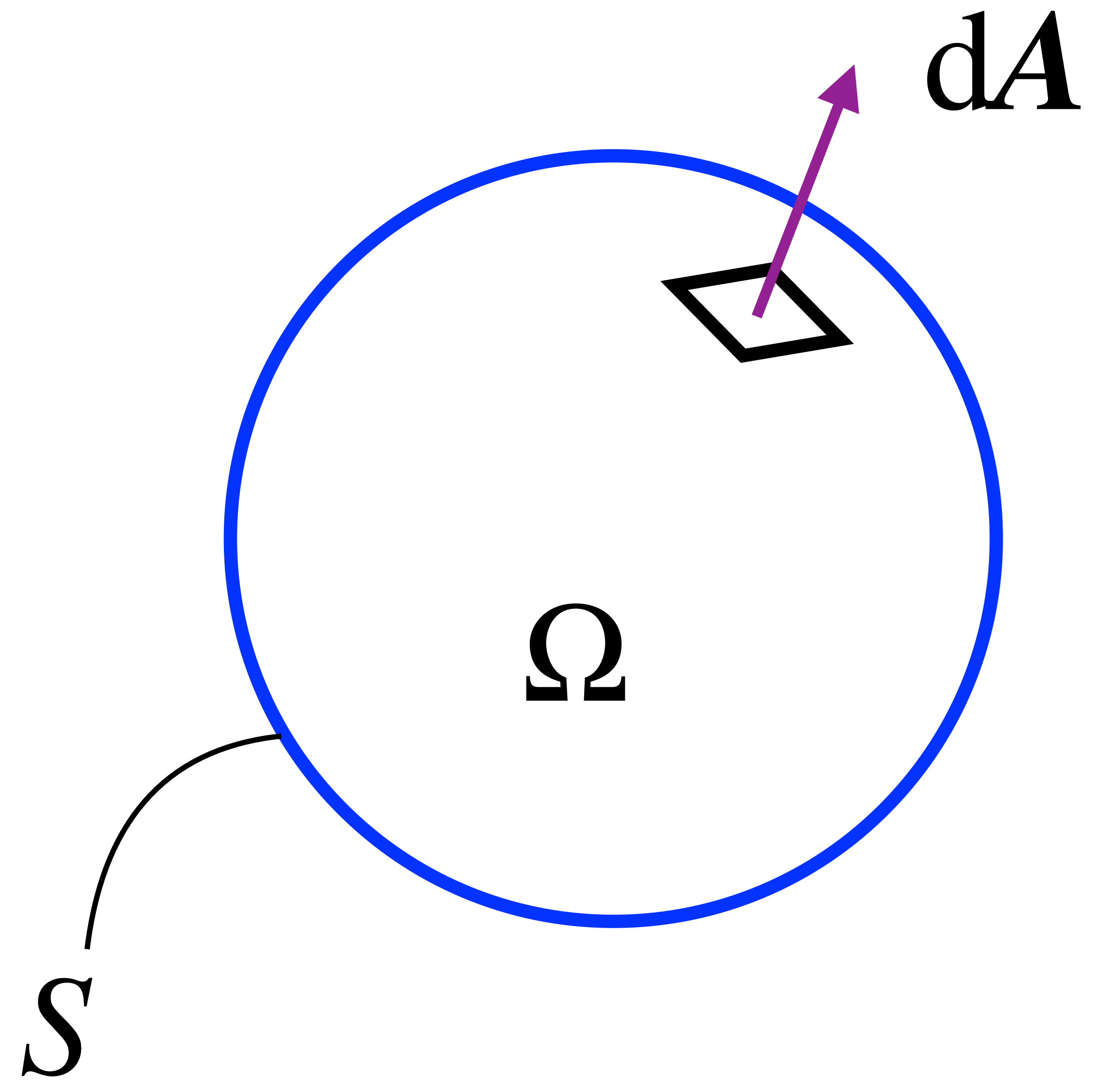
The quantity $\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)$ is called the Lagrangian or convective time derivative.

It's the time derivative in a frame that follows the fluid.

Pressure Force Per Unit Volume $= -\nabla p$



Pressure force on an arbitrary fluid element of volume Ω with boundary S:

$$\boldsymbol{F} = -\oint_S p\,\mathrm{d}\boldsymbol{A} = -\oint_S p\boldsymbol{I}\cdot\mathrm{d}\boldsymbol{A} = -\int_\Omega \nabla\cdot(p\boldsymbol{I})\,\mathrm{d}^3x = -\int_\Omega \nabla p\,\mathrm{d}^3x$$

As Ω is arbitrary, the pressure force per unit volume everywhere is $-\nabla p$

($\boldsymbol{I}$ is the identity matrix. Third equality is Gauss's theorem applied to each component of $\boldsymbol{F}$ separately)

$$\text{pressure force per unit mass} = \frac{\text{pressure force per unit volume}}{\text{mass per unit volume}} = \frac{-\nabla p}{\rho}$$

$\rho =$ mass density of fluid

$$\longrightarrow \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho}$$

Incompressibility

Liquids are hard to compress $\longrightarrow \nabla \cdot \boldsymbol{u} = 0$

Continuity equation then becomes:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla \rho - \cancel{\rho \nabla \cdot \boldsymbol{u}}$$

$$\longrightarrow \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \rho = \frac{\mathrm{d}}{\mathrm{d}t} \rho = 0$$

$\longrightarrow$ If ρ is uniform at $t = 0$, then ρ will be uniform at all t with the same density as at $t = 0$. Henceforth, we will treat ρ as a constant.

Euler Equation

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} = -\nabla\left(\frac{p}{\rho}\right)$$

To simplify notation, we call $\frac{p}{\rho}$ just p. So p is no longer the pressure, but the pressure divided by ρ.

$$\longrightarrow \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p \qquad \text{(Euler equation)}$$

Navier-Stokes Equation: Just Add Viscosity to the Euler Equation

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$$

$\nu =$ (kinematic) viscosity, which we will treat as a constant. The viscosity term is a diffusion term that smoothes out spatial variations in $\boldsymbol{u}$.

Navier-Stokes equation is Newton's 2^{nd} law for a viscous, incompressible fluid.

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How do you determine p given $\boldsymbol{u}$? Take divergence of Navier-Stokes and use $\nabla\cdot\boldsymbol{u} = 0$ to obtain $\nabla^2 p = -\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})$, which is just like the Poisson equation $\nabla^2\Phi = 4\pi\rho_{\text{charge}}$ and can be solved in same way: $\Phi = \int d^3x' \frac{\rho_{\text{charge}}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \longrightarrow p = -\int d^3x' \frac{\nabla'\cdot[\boldsymbol{u}(\boldsymbol{x}',t)\cdot\nabla'\boldsymbol{u}(\boldsymbol{x}',t)]}{|\boldsymbol{x}-\boldsymbol{x}'|}$. Upshot: p becomes whatever it has to be to keep $(\partial/\partial t)(\nabla\cdot\boldsymbol{u}) = 0$ so that $\nabla\cdot\boldsymbol{u}$ continues to vanish.

Energy Conservation in the Inviscid ($\nu = 0$) Limit

Dot $2\boldsymbol{u}$ into: $\quad \dfrac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$

$$\frac{\partial}{\partial t}u^2 + \boldsymbol{u} \cdot \nabla u^2 = -2\boldsymbol{u} \cdot \nabla p$$

because $\dfrac{\partial}{\partial t}(\boldsymbol{u} \cdot \boldsymbol{u}) = 2\boldsymbol{u} \cdot \dfrac{\partial \boldsymbol{u}}{\partial t}$

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$$\int_\Omega \left(\frac{\partial}{\partial t}u^2\right)\mathrm{d}^3x + \oint_{\partial\Omega} u^2\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A} = -2\oint_{\partial\Omega} p\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A}$$

Integrate over some region Ω with boundary $\partial\Omega$. Use Gauss's theorem on divergence terms.

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^2\,\mathrm{d}^3x\right) = 0 \longrightarrow \text{`Energy'} = \int_\Omega u^2\,\mathrm{d}^3x \text{ is conserved}$$

really twice the bulk-flow kinetic energy divided by ρ

Take Ω to be all space and the fluid to be confined to a finite volume $\rightarrow$ boundary terms vanish

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Fourier Transforms

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \text{Fourier transform operator}$$

$$\tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t) = \mathscr{F}(\boldsymbol{u}(\boldsymbol{x}, t))$$

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We're basically writing $\boldsymbol{u}(\boldsymbol{x},t)$ as a weighted 'sum' (integral) of plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \cos(\boldsymbol{k}\cdot\boldsymbol{x}) + i\sin(\boldsymbol{k}\cdot\boldsymbol{x})$. We need to assume that $\boldsymbol{u}(\boldsymbol{x},t)$ vanishes as $|\boldsymbol{x}| \to \infty$.

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Apply $\mathscr{F}$ to $\quad \dfrac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u} \quad$ to obtain

$$\frac{\partial \tilde{\boldsymbol{u}}_{\boldsymbol{k}}}{\partial t} + \frac{i}{(2\pi)^{3/2}} \int \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}^3\boldsymbol{q}\, (\tilde{\boldsymbol{u}}_{\boldsymbol{p}}\cdot\boldsymbol{q})\, \tilde{\boldsymbol{u}}_{\boldsymbol{q}}\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) = -i\boldsymbol{k}\tilde{p}_k - k^2\nu\,\tilde{\boldsymbol{u}}_{\boldsymbol{k}}$$

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$\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}$ are wave vectors. k is like $\frac{1}{\text{wavelength}}$ or $\frac{1}{\text{eddy size}}$.

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$ —- Fourier transform operator

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$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\partial}{\partial x}\left(e^{-ik_x x}\boldsymbol{u}\right) - \boldsymbol{u}\frac{\partial}{\partial x} e^{-ik_x x}\right]$$

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$ —- Fourier transform operator

$$\mathscr{F}(\boldsymbol{u}) = \tilde{\boldsymbol{u}}_{\boldsymbol{k}}$$

$$\mathscr{F}\left(\frac{\partial \boldsymbol{u}}{\partial x}\right) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-i(k_x x + k_y y + k_z z)} \frac{\partial \boldsymbol{u}}{\partial x}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ik_x x} \frac{\partial \boldsymbol{u}}{\partial x}$$

total-derivative term vanishes because $\boldsymbol{u}$ vanishes as $x \to \pm\infty$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\partial}{\partial x}\left(e^{-ik_x x} \boldsymbol{u}\right) - \boldsymbol{u}\frac{\partial}{\partial x} e^{-ik_x x}\right]$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \left[-\boldsymbol{u}(-ik_x) e^{-ik_x x}\right] = ik_x \tilde{\boldsymbol{u}}_{\boldsymbol{k}}$$

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$ —- Fourier transform operator

$$\mathscr{F}(\boldsymbol{u}) = \tilde{\boldsymbol{u}}_{\boldsymbol{k}} \qquad \mathscr{F}\left(\frac{\partial \boldsymbol{u}}{\partial x}\right) = ik_x \tilde{\boldsymbol{u}}_{\boldsymbol{k}}$$

Under Fourier transform, $\frac{\partial}{\partial x} \longrightarrow ik_x$ and $\nabla \longrightarrow i\boldsymbol{k}$

$$\mathscr{F}\left(\frac{\partial \boldsymbol{u}}{\partial x}\right) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-i(k_x x + k_y y + k_z z)} \frac{\partial \boldsymbol{u}}{\partial x}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ik_x x} \frac{\partial \boldsymbol{u}}{\partial x}$$

total-derivative term vanishes because $\boldsymbol{u}$ vanishes as $x \to \pm\infty$

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$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-ik_y y} \int_{-\infty}^{\infty} \mathrm{d}z \, e^{-ik_z z} \int_{-\infty}^{\infty} \mathrm{d}x \left[-\boldsymbol{u}(-ik_x) e^{-ik_x x}\right] = ik_x \tilde{\boldsymbol{u}}_{\boldsymbol{k}}$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{k}}$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})$$

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$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ldots.$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \qquad f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{k}}$$

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3p \, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}} \; \ldots .$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})$$

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$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3p \, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3q \, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \tilde{g}_{\boldsymbol{q}}$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \qquad f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{k}}$$

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 p \, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 q \, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \tilde{g}_{\boldsymbol{q}}$$

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{9/2}} \int \mathrm{d}^3 x \, \mathrm{d}^3 p \, \mathrm{d}^3 q \, e^{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}} \tilde{g}_{\boldsymbol{q}}$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \qquad f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{k}}$$

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$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{9/2}} \int \mathrm{d}^3x\, \mathrm{d}^3p\, \mathrm{d}^3q\, e^{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}}\, \tilde{g}_{\boldsymbol{q}}$$

Use $\int \mathrm{d}^3x\, e^{i\boldsymbol{x}\cdot\boldsymbol{w}} = (2\pi)^3 \delta(\boldsymbol{w})$ (exercise: show this. Requires complex analysis…)

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \qquad f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{k}}$$

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$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{9/2}} \int \mathrm{d}^3x \, \mathrm{d}^3p \, \mathrm{d}^3q \, e^{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \tilde{f}_{\boldsymbol{p}} \tilde{g}_{\boldsymbol{q}}$$

Use $\int \mathrm{d}^3x \, e^{i\boldsymbol{x}\cdot\boldsymbol{w}} = (2\pi)^3 \delta(\boldsymbol{w})$ (exercise: show this. Requires complex analysis…)

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3p \, \mathrm{d}^3q \, \tilde{f}_{\boldsymbol{p}} \tilde{g}_{\boldsymbol{q}} \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q})$$

Convolution Theorem

$$\tilde{f}_{\boldsymbol{k}} = \mathscr{F}(f(\boldsymbol{x})) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \qquad f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{f}_{\boldsymbol{k}}$$

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3p\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\, \tilde{f}_{\boldsymbol{p}} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3q\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\, \tilde{g}_{\boldsymbol{q}}$$

$$\mathscr{F}(f(x)g(x)) = \frac{1}{(2\pi)^{9/2}} \int \mathrm{d}^3x\, \mathrm{d}^3p\, \mathrm{d}^3q\, e^{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}}\, \tilde{f}_{\boldsymbol{p}}\, \tilde{g}_{\boldsymbol{q}}$$

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Revisit Previous slide on Fourier Transforms

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$ = Fourier transform operator

$$\mathscr{F}^{-1} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\tilde{\boldsymbol{u}}_k(t) = \mathscr{F}(\boldsymbol{u}(\boldsymbol{x},t)) \qquad \boldsymbol{u}(\boldsymbol{x},t) = \mathscr{F}^{-1}(\tilde{\boldsymbol{u}}_k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \tilde{\boldsymbol{u}}_k(t)$$

We're basically writing $\boldsymbol{u}(\boldsymbol{x},t)$ as a weighted 'sum' (integral) of plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \cos(\boldsymbol{k}\cdot\boldsymbol{x}) + i\sin(\boldsymbol{k}\cdot\boldsymbol{x})$. We need to assume that $\boldsymbol{u}(\boldsymbol{x},t)$ vanishes as $|\boldsymbol{x}| \to \infty$.

Apply $\mathscr{F}$ to $\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$ to obtain

$$\frac{\partial \tilde{\boldsymbol{u}}_k}{\partial t} + \frac{i}{(2\pi)^{3/2}} \int \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}^3\boldsymbol{q}\, (\tilde{\boldsymbol{u}}_p \cdot \boldsymbol{q})\, \tilde{\boldsymbol{u}}_q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) = -i\boldsymbol{k}\tilde{p}_k - k^2\nu\, \tilde{\boldsymbol{u}}_k$$

$\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}$ are wave vectors. k is like $\frac{1}{\text{wavelength}}$ or $\frac{1}{\text{eddy size}}$.

$$\int \mathrm{d}^3x \; \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \int \mathrm{d}^3k \int \mathrm{d}^3k' \, e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')} \, \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t) \cdot \tilde{\boldsymbol{u}}_{\boldsymbol{k}'}(t)$$

The Energy Spectrum

$$\int \mathrm{d}^3x\, \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x \int \mathrm{d}^3k \int \mathrm{d}^3k'\, e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')}\, \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t)\cdot\tilde{\boldsymbol{u}}_{\boldsymbol{k}'}(t)$$

The Energy Spectrum

Use $\int \mathrm{d}^3x\, e^{i\boldsymbol{x}\cdot\boldsymbol{w}} = (2\pi)^3\delta(\boldsymbol{w})$

$$\int d^3x\, \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int d^3x \int d^3k \int d^3k'\, e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')}\, \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t)\cdot\tilde{\boldsymbol{u}}_{\boldsymbol{k}'}(t)$$

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$$\int \mathrm{d}^3x\, \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x \int \mathrm{d}^3k \int \mathrm{d}^3k'\, e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')}\, \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t)\cdot\tilde{\boldsymbol{u}}_{\boldsymbol{k}'}(t)$$

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Note: $\tilde{u}_{-\boldsymbol{k}}(t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\, e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x},t) = \left[\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x},t)\right]^* = \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t)^*$

The Energy Spectrum

$$\int d^3x\, \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int d^3x \int d^3k \int d^3k'\, e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')}\,\tilde{\boldsymbol{u}}_{\boldsymbol{k}}(t)\cdot\tilde{\boldsymbol{u}}_{\boldsymbol{k}'}(t)$$

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$$\langle u^2\rangle = \frac{1}{V}\int d^3x\,|\boldsymbol{u}(\boldsymbol{x},t)|^2 = \frac{1}{V}\int d^3k\,|\tilde{\boldsymbol{u}}_{\boldsymbol{k}}|^2 = \frac{4\pi}{V}\int_0^\infty dk\, k^2|\tilde{\boldsymbol{u}}_{\boldsymbol{k}}|^2 = \int_0^\infty dk\, E(k)$$

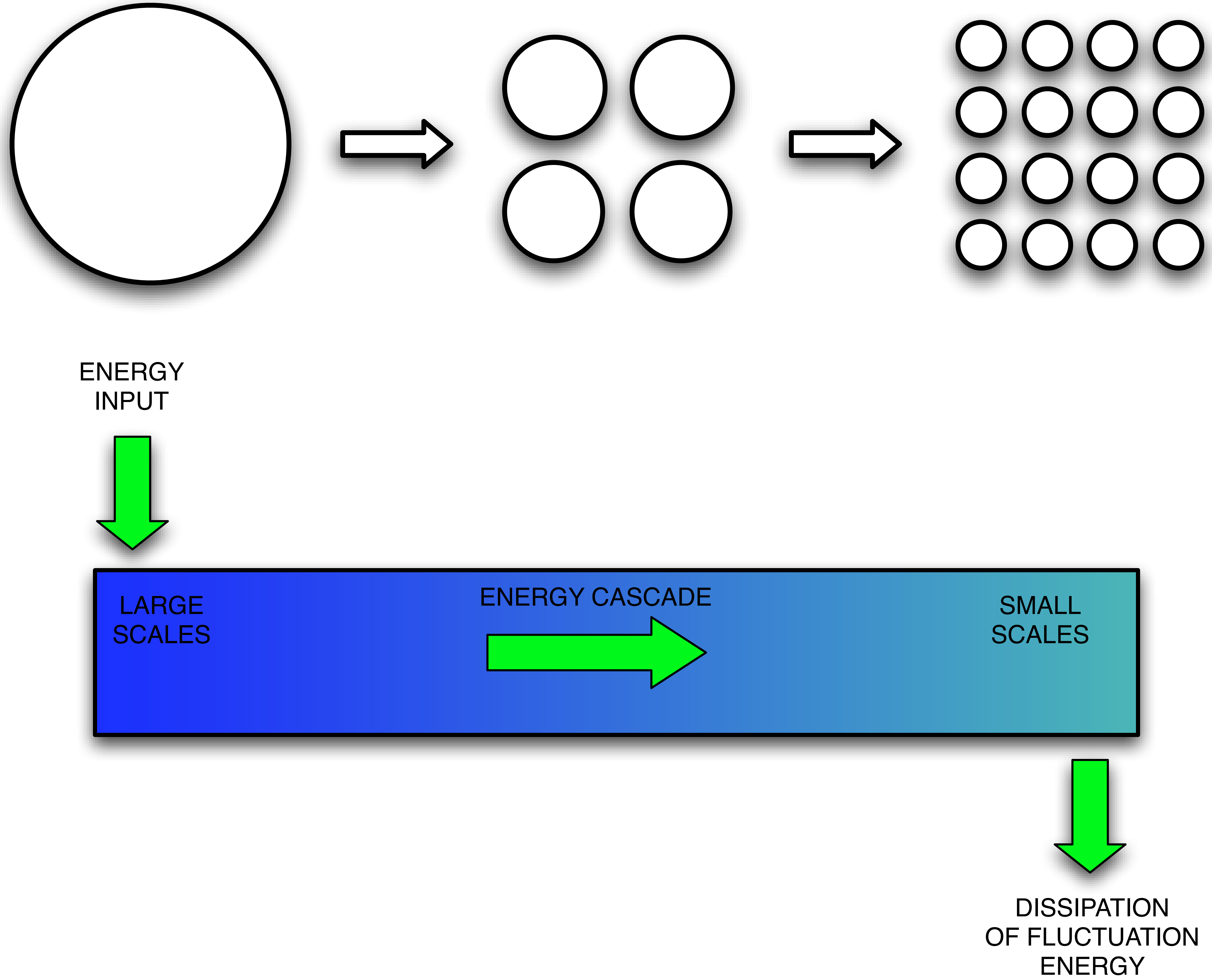
where $E(k) = 4\pi k^2|\tilde{\boldsymbol{u}}_{\boldsymbol{k}}|^2/V$ is the 'energy spectrum' or 'power spectrum,' and V is the fluid volume. I've assumed isotropy — that $|\tilde{\boldsymbol{u}}_{\boldsymbol{k}}|^2$ is independent of the direction of $\boldsymbol{k}$.

Outline

1. Intro: what is turbulence?
2. Review: continuity equation, Navier-Stokes equation
3. Review: Fourier transforms
4. Energy cascade from large scales to small scales in 3D hydro
5. Inverse energy cascade from small scales to large scales in 2D hydro
6. Extras: turbulent transport, turbulent heating, passive-scalar diffusion

Energy Cascade

Canonical picture: larger eddies break up into smaller eddies



How Important is the Viscous Term in the Navier-Stokes Equation?

Navier-Stokes equation:
$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}$$

Let turbulence be initiated by stirring eddies with characteristic diameter L (the ‘forcing scale’ or ‘outer scale’) and characteristic velocity u_L. (The terms on the left are called the ‘inertial terms’.)

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‘Eddy turnover time’ at scale L is $\tau_L = \frac{L}{u_L}$, and $\frac{\partial}{\partial t} \sim \frac{1}{\tau_L} \sim \frac{u_L}{L}$

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Inertial terms: $\dfrac{\partial\boldsymbol{u}}{\partial t} \sim \dfrac{u_L^2}{L}$, $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \sim \dfrac{u_L^2}{L}$

Viscous term: $\nu\nabla^2\boldsymbol{u} \sim \dfrac{\nu u_L}{L^2}$

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Note: because of the ∇^2, the viscous term becomes more important at smaller scales

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Viscous term: $$\nu\nabla^2\boldsymbol{u} \sim \frac{\nu u_L}{L^2}$$

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Viscous term: $\nu\nabla^2\boldsymbol{u} \sim \frac{\nu u_L}{L^2}$

$$\frac{\text{inertial terms}}{\text{viscous term}} \sim \frac{u_L^2/L}{\nu u_L/L^2} = \frac{u_L L}{\nu} \equiv \text{the Reynolds number} \equiv \text{Re}$$

$\leftarrow$ dimensionless because units of ν are $\frac{\text{length}^2}{\text{time}}$

How Important is the Viscous Term in the Navier-Stokes Equation?

Navier-Stokes equation: $$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$$

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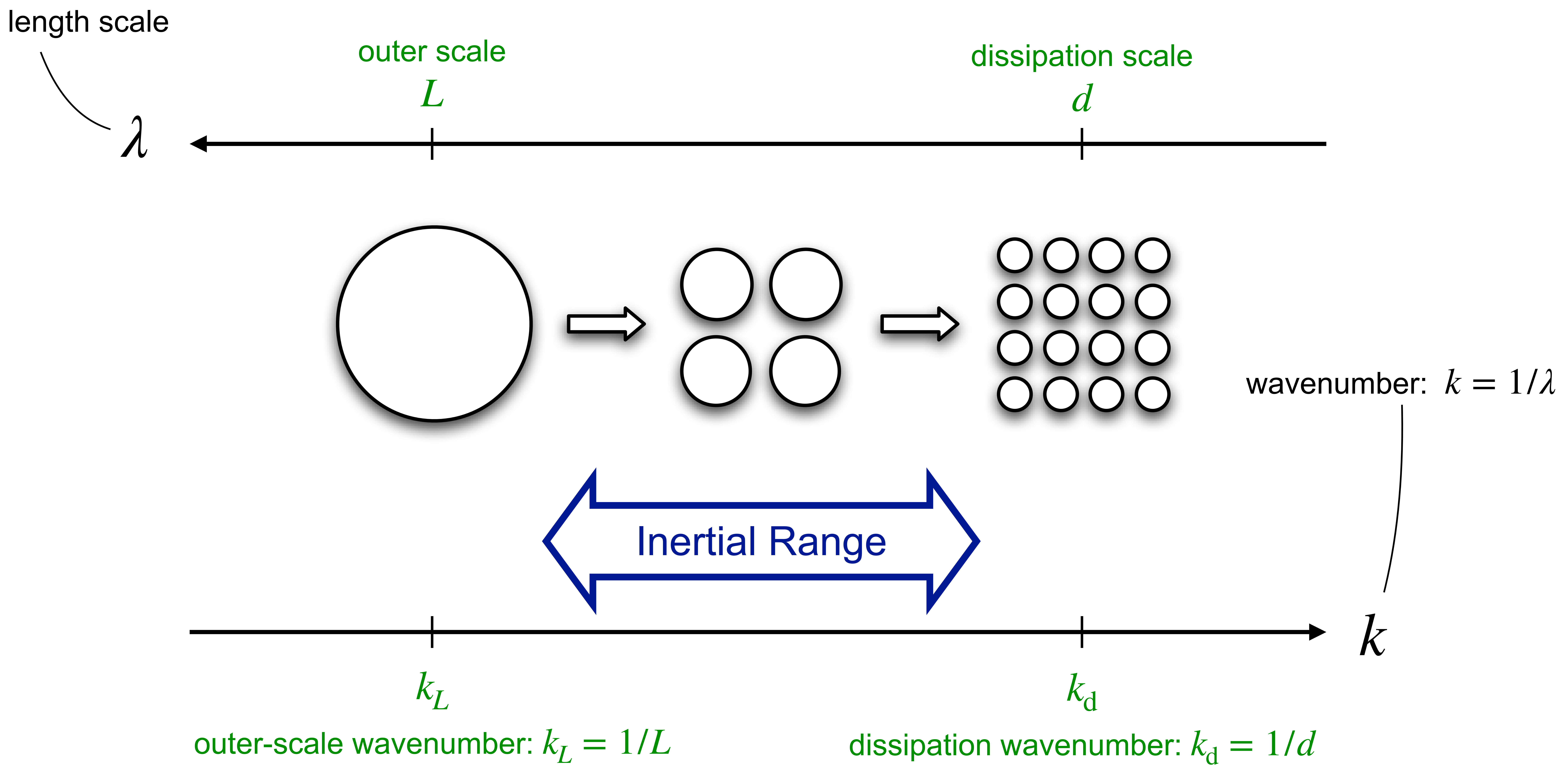
$$\frac{\text{inertial terms}}{\text{viscous term}} \sim \frac{u_L^2/L}{\nu u_L/L^2} = \frac{u_L L}{\nu} \equiv \text{the Reynolds number} \equiv \mathrm{Re}$$ ← dimensionless because units of ν are $\frac{\text{length}^2}{\text{time}}$

When $\mathrm{Re} \gg 1$, viscosity is unimportant at the outer scale. Outer-scale eddies then 'turn over' freely and break up into smaller eddies. ⟶ turbulence arises at large Reynolds number. We will henceforth assume $\mathrm{Re} \gg 1$.

When $\mathrm{Re} \lesssim 1$, there is no turbulence, and the flow is called `laminar.'



'Inertial range' = scales λ satisfying $d \ll \lambda \ll L$, or, equivalently, wavenumbers k satisfying $k_L \ll k \ll k_\mathrm{d}$

Inertial Range: $d \ll \lambda \ll L \longleftrightarrow k_L \ll k \ll k_{\mathrm{d}}$

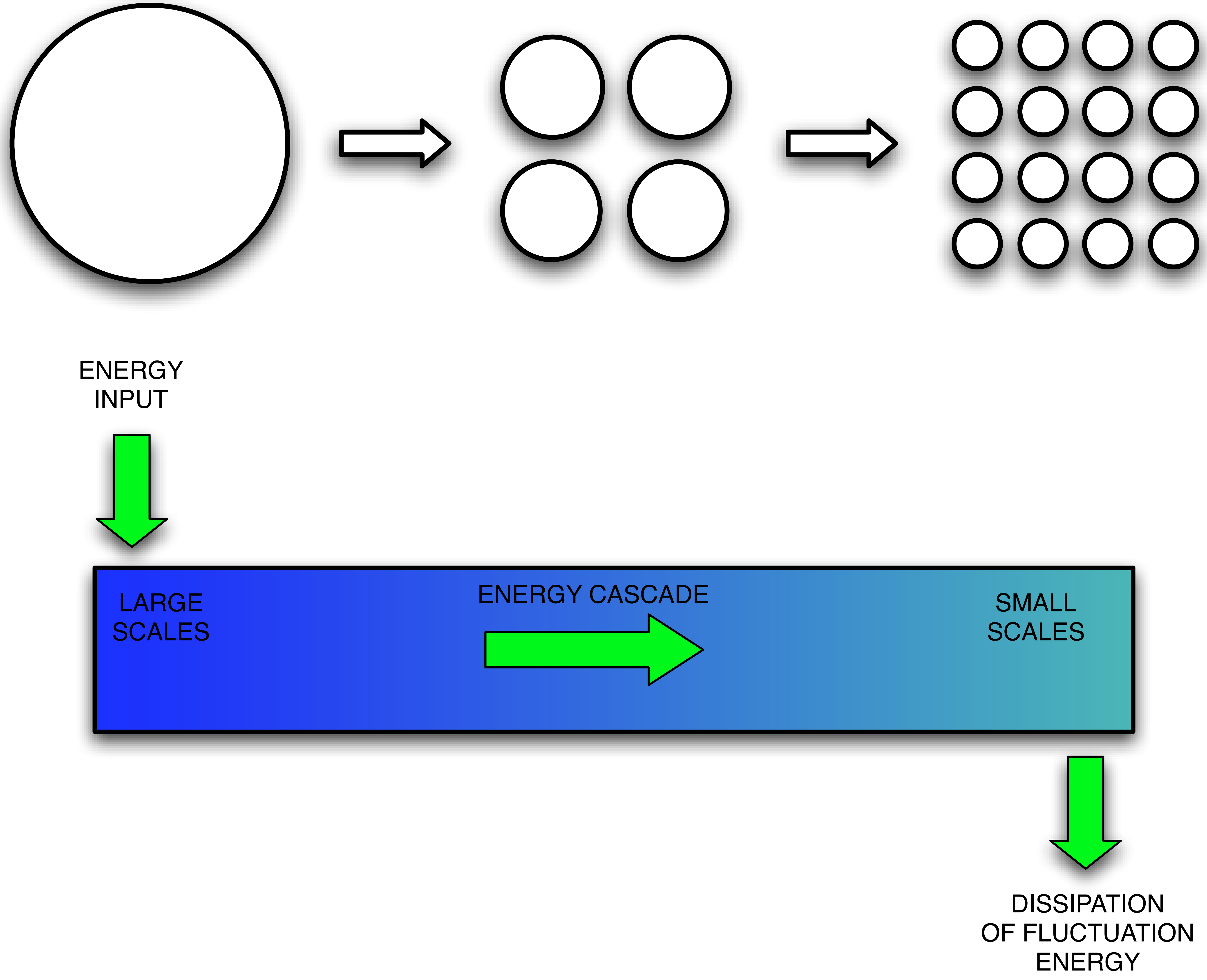
- $\lambda \ll L$ means that the dynamics at scale λ are not influenced by the details of the forcing at scale L

- $\lambda \gg d$ means that the dynamics at scale λ are not influenced by dissipation (viscosity)

- Because the dynamics in the inertial range are independent of the forcing and dissipation, they are plausibly universal — i.e., independent of exactly how you set up the turbulence. Inertial range is also approximately isotropic and homogeneous.

Energy Cascade

Canonical picture: larger eddies break up into smaller eddies



Turbulence Requires Nonlinearity

Navier-Stokes equation: $$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$$

Drop nonlinear term: $$\frac{\partial}{\partial t}\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$$

Take curl and define the vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$: $$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nu\nabla^2\boldsymbol{\omega}$$

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Aside: this is a diffusion equation. Solve by taking Fourier transform:

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}} = -k^2\nu\,\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}} \longrightarrow \tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(t) = \tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(0)\,e^{-k^2\nu t} \qquad \boldsymbol{\omega}(\boldsymbol{x},t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\,\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(0)\,e^{-k^2\nu t + i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Key point: viscosity damps Fourier modes at a rate $k^2\nu$ that becomes large when k gets large (i.e., at small scales). This quantifies our earlier statement that viscous damping is strong at small scales.

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Linear equation $\longrightarrow$ sum of two solutions is also a solution $\longrightarrow$ if you had multiple “eddy” solutions, they would not interact, but would instead pass through one another unchanged.

But the nonlinear $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term gives rise to interactions between eddies — these are called ’nonlinear interactions’.

Recall from Earlier Slide on Fourier Transforms

$$\mathscr{F} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \text{Fourier transform operator} \qquad \mathscr{F}^{-1} = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\tilde{\boldsymbol{u}}_k(t) = \mathscr{F}(\boldsymbol{u}(\boldsymbol{x},t)) \qquad \boldsymbol{u}(\boldsymbol{x},t) = \mathscr{F}^{-1}(\tilde{\boldsymbol{u}}_k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{\boldsymbol{u}}_k(t)$$

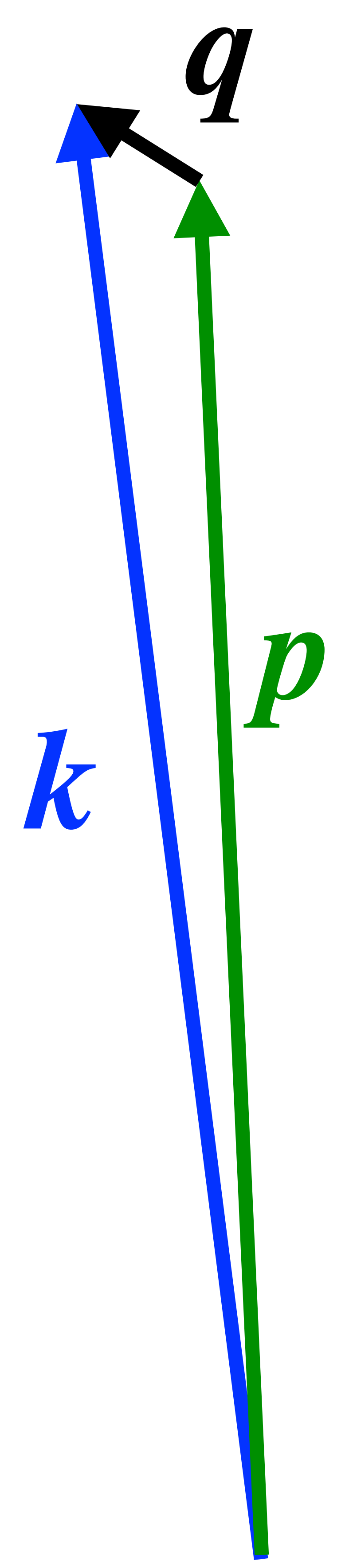
We're basically writing $\boldsymbol{u}(\boldsymbol{x},t)$ as a weighted 'sum' (integral) of plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \cos(\boldsymbol{k}\cdot\boldsymbol{x}) + i\sin(\boldsymbol{k}\cdot\boldsymbol{x})$. We need to assume that $\boldsymbol{u}(\boldsymbol{x},t)$ vanishes as $|\boldsymbol{x}| \to \infty$.

Apply $\mathscr{F}$ to $\quad \dfrac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u} \quad$ to obtain

$$\frac{\partial \tilde{\boldsymbol{u}}_k}{\partial t} + \frac{i}{(2\pi)^{3/2}} \int \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}^3\boldsymbol{q}\, (\tilde{\boldsymbol{u}}_p \cdot \boldsymbol{q})\, \tilde{\boldsymbol{u}}_q\, \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}) = -i\boldsymbol{k}\tilde{p}_k - k^2\nu\, \tilde{\boldsymbol{u}}_k$$

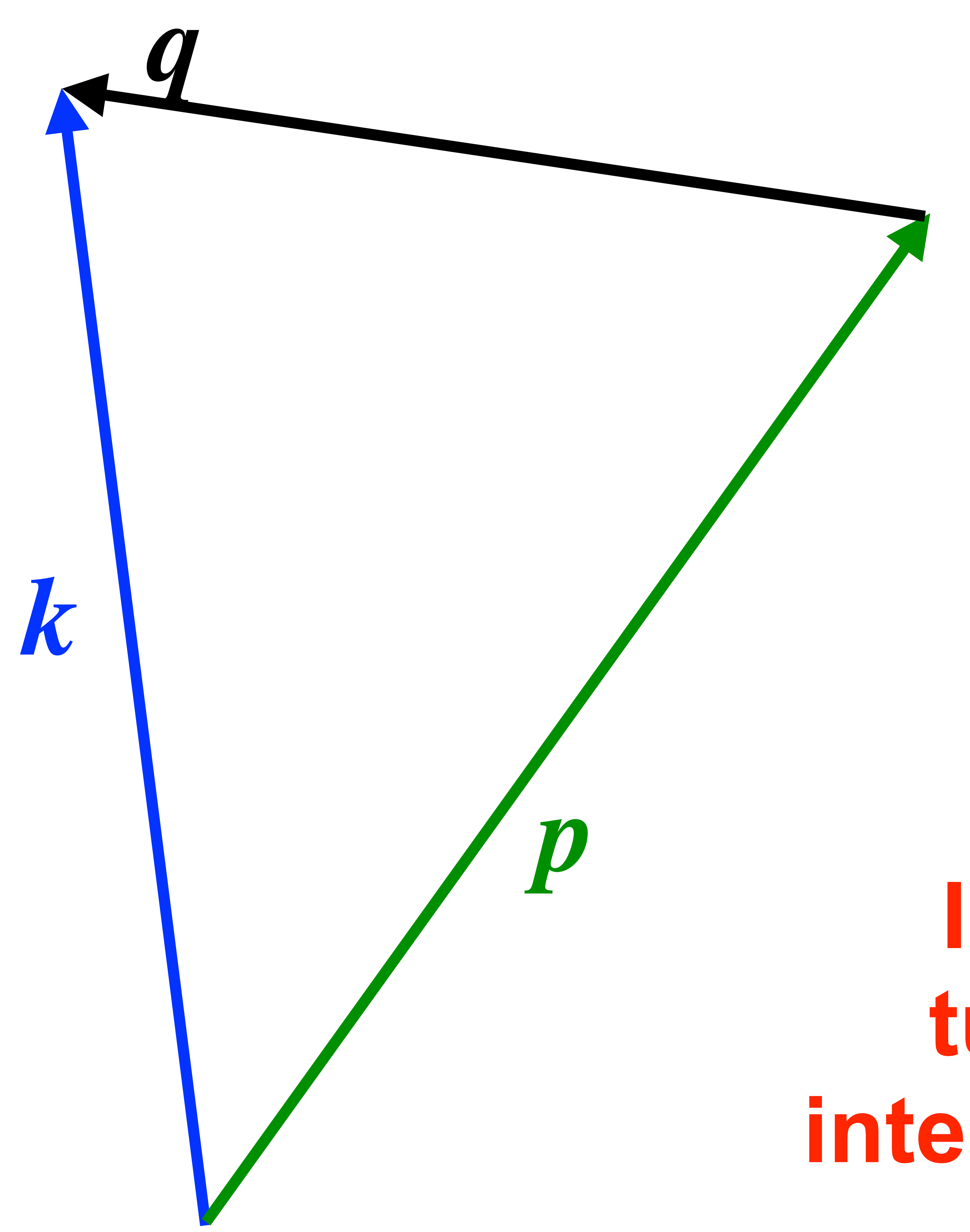
Three Fourier modes $\tilde{\boldsymbol{u}}_k$, $\tilde{\boldsymbol{u}}_p$, and $\tilde{\boldsymbol{u}}_q$ can only interact if interact $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$.

Wave Vector Triads



$q \ll k \sim p$

"nonlocal interactions"



$k \sim p \sim q$

"local interactions"

In most types of turbulence, local interactions dominate.

Scale-Dependent Velocity-Fluctuation Amplitude u_λ or u_k

$u_\lambda \equiv$ rms amplitude of velocity fluctuations (eddies) with length scales $\sim \lambda$

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Note the notational confusion: u_λ is not the same thing as "u_k evaluated at $k = \lambda$."

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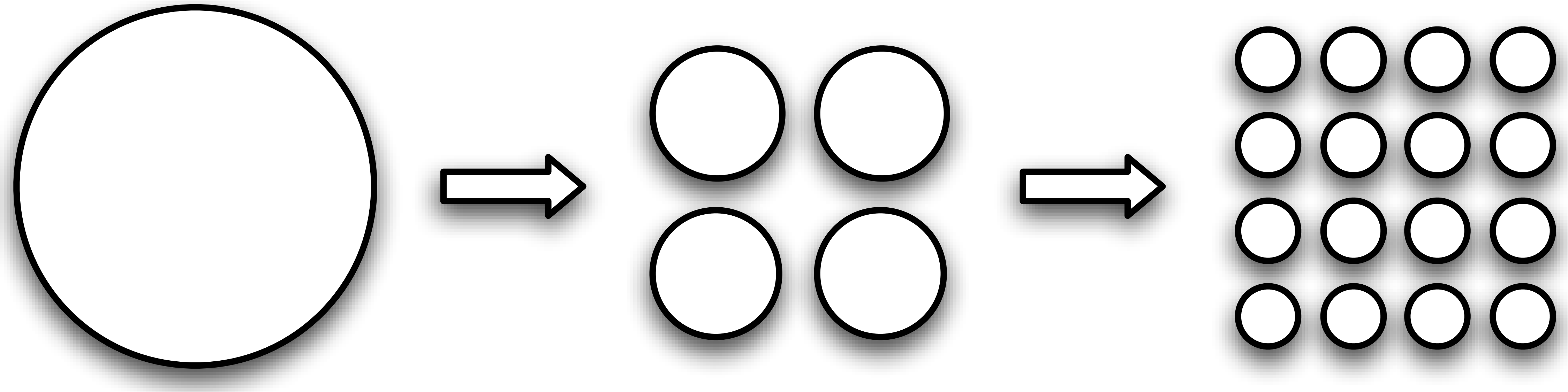
From before: $\langle u^2 \rangle = \int_0^\infty E(k)\,\mathrm{d}k$

$\longrightarrow u_k^2 \sim \int_{k/2}^{2k} E(k')\,\mathrm{d}k'$ — i.e., u_k^2 is the contribution to $\langle u^2 \rangle$ from all wavenumbers k' that are within a factor of ~ 2 of k

$\longrightarrow u_k^2 \sim kE(k) \qquad u_k \sim \left[kE(k)\right]^{1/2} \qquad u_\lambda \sim \left[kE(k)\right]^{1/2}_{k=1/\lambda}$

Kolmogorov Energy Spectrum

(Kolmogorov 1941)



- In the inertial range, eddies of scale λ break up on their turnover time $\tau_\lambda \sim \frac{\lambda}{u_\lambda}$ and pass their energy $\sim u_\lambda^2$ on to smaller eddies.
- This sets up an energy flux ϵ *in wavenumber space* from small k to large k, or, equivalently, from large λ to small λ, where $\epsilon \sim \frac{u_\lambda^2}{\tau_\lambda} = \frac{u_\lambda^3}{\lambda}$
- ϵ must be independent of λ in the inertial range, where forcing and dissipation play no role, because local interactions dominate (i.e., eddies of size λ break up into eddies of size $\sim \lambda/2$ which later break up into eddies of size $\sim \lambda/4$)
- $\longrightarrow u_\lambda \propto \lambda^{1/3} \longrightarrow kE(k) = (u_\lambda^2)_{\lambda=1/k} \propto (\lambda^{2/3})_{\lambda=1/k} = k^{-2/3} \qquad \longrightarrow E(k) \propto k^{-5/3}$

Scale Dependence of Eddy Turnover Time

$u_\lambda \propto \lambda^{1/3} \propto k^{-1/3}$ in inertial range

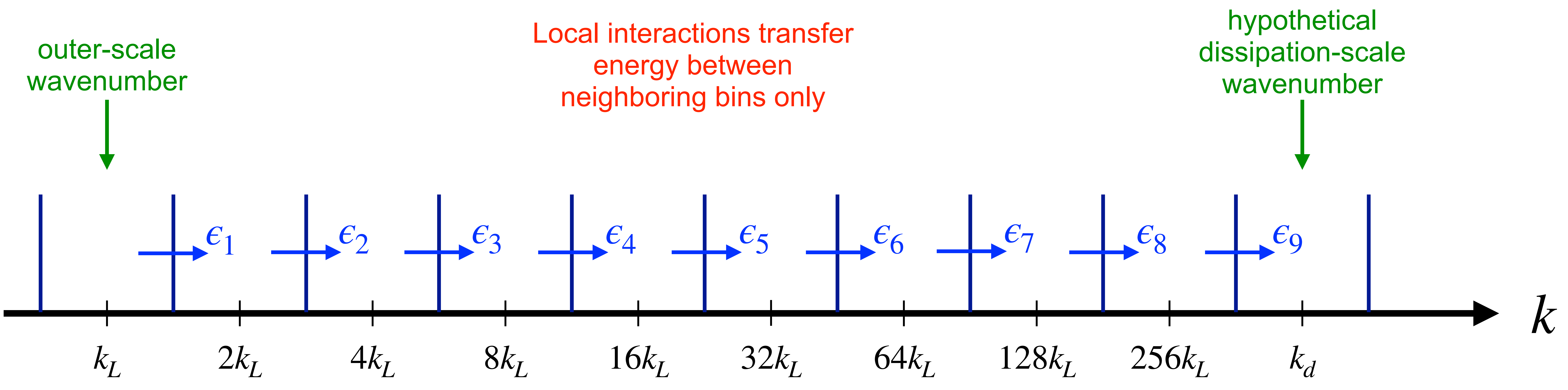
$\tau_\lambda = \dfrac{\lambda}{u_\lambda} \propto \lambda^{2/3} \longrightarrow$ eddy turnover time is smaller for smaller-scale eddies, or, equivalently:

$\dfrac{1}{\tau_\lambda} \propto k^{2/3} \longrightarrow$ eddy turnover rate is larger at higher k

As $\mathrm{Re} \to \infty$, eddies deep within the inertial range turn over infinitely faster than the outer-scale eddies. Thus, even if $E(k)$ changes on the dynamical time scale of the outer-scale eddies, $E(k)$ should be in steady state on the dynamical time scale(s) of the inertial range eddies.

Also, in ~ 2 outer-scale turnover times ($2L/u_L$) energy cascades all the way to the dissipation scale d, no matter how small d is. (Total time needed is a geometric series like $1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots$)

So Why Again Is ϵ Constant?



If $\epsilon_5 > \epsilon_6$, then energy builds up in the wavenumber bin centered on $32k_L$. But this is impossible at large Re, because we just concluded that $E(k)$ should be in steady state on the time scales of the inertial-range eddies.

Therefore, $\epsilon_5 = \epsilon_6$, and more generally ϵ is independent of k within inertial range at large Re.

Dissipation Scale or Kolmogorov Scale

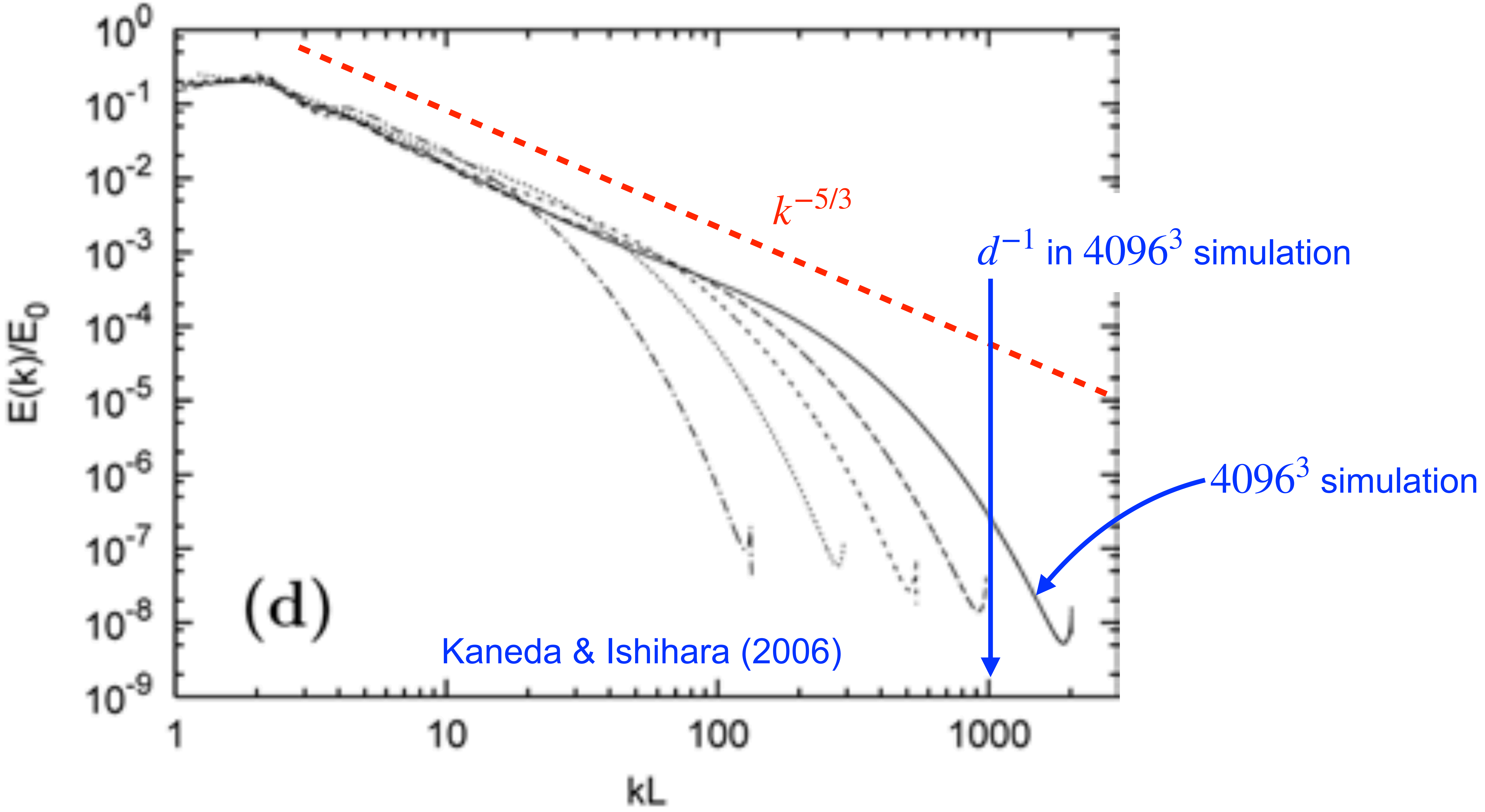
$$u_\lambda \sim U_L \left(\frac{\lambda}{L}\right)^{1/3}$$ in inertial range.

At dissipation scale, local-eddy shearing rate is comparable to viscous damping rate: $\frac{u_\lambda}{\lambda} \sim \frac{\nu}{\lambda^2} \longrightarrow \frac{u_\lambda \lambda}{\nu} \sim \mathrm{O}(1) \longrightarrow \frac{u_L L}{\nu}\left(\frac{\lambda}{L}\right)^{4/3} \sim \mathrm{O}(1)$

$$\longrightarrow \lambda \sim \mathrm{Re}^{-3/4} L$$

The Kolmogorov scale or dissipation scale $d = \mathrm{Re}^{-3/4} L$ is the scale at which viscous damping would be as efficient as inertial-range cascading. This is effectively the scale at which viscosity truncates the cascade. As $\mathrm{Re} \to \infty$, $d \to 0$.

Numerical Simulations of 3D Hydrodynamic Turbulence



Five simulations with, respectively, 256^3, 512^3, 1024^3, 2048^3, and 4096^3 grid points. As the authors increased their numerical resolution, they simultaneously decreased the viscosity, causing the inertial range to broaden.

Snapshot of Intense Vorticity Structures

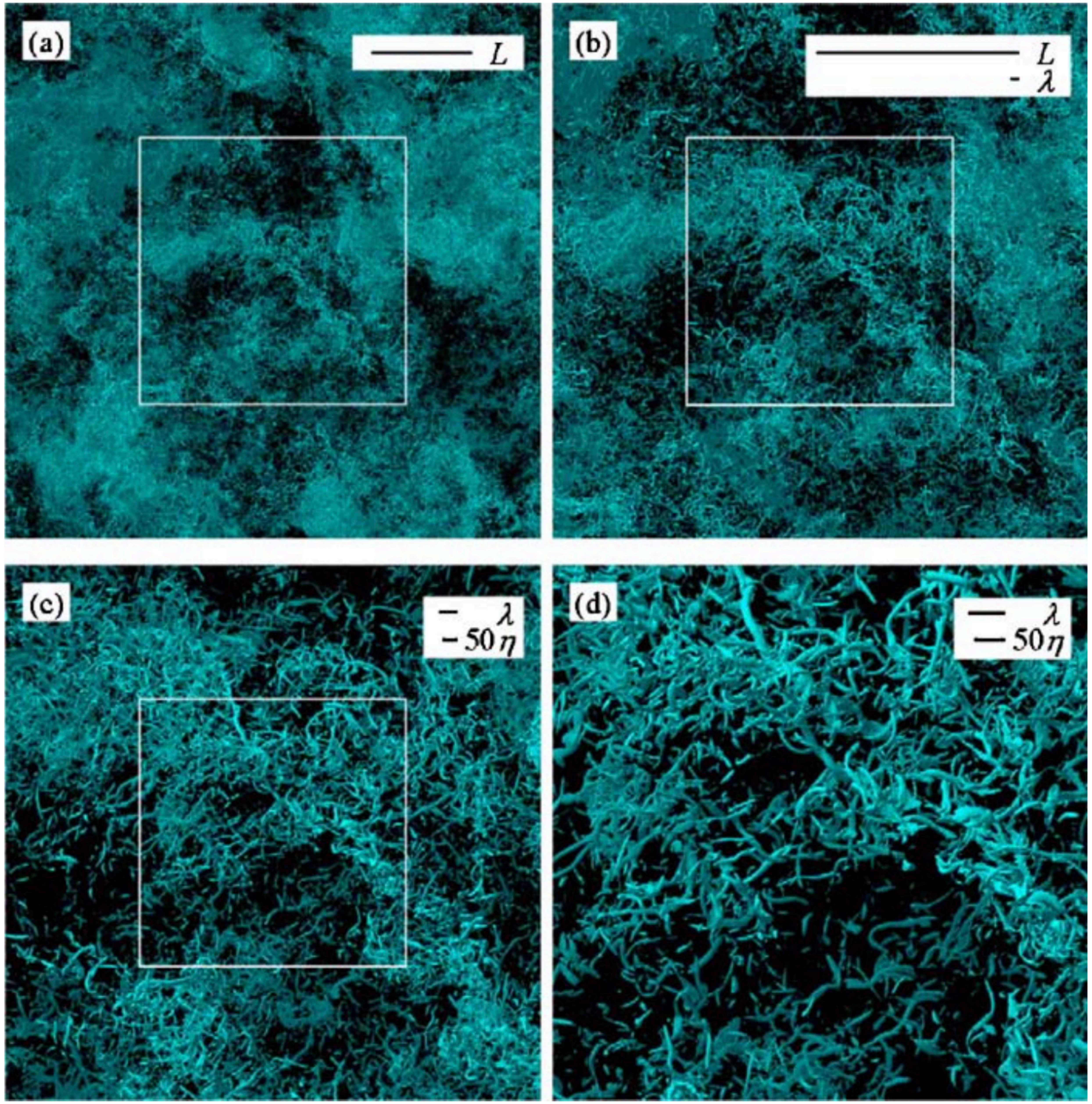


Figure 3. Intense-vorticity isosurfaces showing the region where $\omega > \langle\omega\rangle + 4\sigma_\omega$. $R_\lambda = 732$. (a) The size of the display domain is $(5984^2 \times 1496)\,\eta^3$, periodic in the vertical and horizontal directions. (b) Close-up view of the central region of (a) bounded by the white rectangular line; the size of display domain is $(2992^2 \times 1496)\,\eta^3$. (c) Close-up view of the central region of (b); $1496^3\eta^3$ (d) Close-up view of the central region of (c); $(748^2 \times 1496)\,\eta^3$.

Vorticity = $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

Starting in panel (a), each successive panel (b, c, d) zooms in on the central region of the previous panel

Kaneda & Ishihara (2006)

Zeroth Law of Turbulence

As $\nu \to 0$, the dissipation scale gets smaller, but the energy cascade rate is independent of ν.

The energy cascade rate in high-Reynolds number (Re) turbulence can be estimated just from knowledge of the outer-scale eddies, and it is just u_L^3/L.

Outline

1. Intro: what is turbulence?
2. Review: continuity equation, Navier-Stokes equation
3. Review: Fourier transforms
4. Energy cascade from large scales to small scales in 3D hydro
5. Inverse energy cascade from small scales to large scales in 2D hydro
6. Extras: turbulent transport, turbulent heating, passive-scalar diffusion

$$\nabla \times \left\{ \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \right\}$$

2D Hydro

$$\nabla \times \left\{ \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \right\}$$

2D Hydro

Vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Note: $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla \frac{u^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega}$ because:

recall: ϵ_{ijk} is the Levi-Civita symbol, and $(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k$

$\epsilon_{ijk} = 1$ if $(i, j, k) = (1,2,3), (2,3,1)$, or $(3,1,2)$

$\epsilon_{ijk} = -1$ if $(i, j, k) = (2,1,3), (3,2,1)$, or $(1,3,2)$

$\epsilon_{ijk} = 0$ otherwise; i.e., if any two indices are the same

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$$[\boldsymbol{u} \times (\nabla \times \boldsymbol{u})]_i = \epsilon_{ijk} u_j (\epsilon_{klm} \partial_l u_m) = \epsilon_{kij} \epsilon_{klm} u_j \partial_l u_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \partial_l u_m = \left[\nabla \frac{u^2}{2} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right]_i$$

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Aside: let's show that $\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Einstein summation convention: sum over repeated indices: $\epsilon_{kij}\epsilon_{klm} \equiv \sum_k \epsilon_{kij}\epsilon_{klm}$

$$\epsilon_{ijk} = 1 \text{ if } (i,j,k) = (1,2,3),\ (2,3,1),\ \text{or } (3,1,2)$$
$$\epsilon_{ijk} = -1 \text{ if } (i,j,k) = (2,1,3),\ (3,2,1),\ \text{or } (1,3,2)$$
$$\epsilon_{ijk} = 0 \text{ otherwise; i.e., if any two indices are the same}$$

If $(i, j) = (l, m)$ and (i, j) are two different numbers, then $\epsilon_{kij}\epsilon_{klm} = (\pm 1)(\pm 1) = 1$

e.g., if $(i, j) = (l, m) = (2,3)$, then $\epsilon_{kij}\epsilon_{klm} = \epsilon_{123}\epsilon_{123} + \epsilon_{223}\epsilon_{223} + \epsilon_{323}\epsilon_{323} = 1 \times 1 + 0 + 0 = 1$

,

If $(i,j)=(l,m)$ and (i,j) are two different numbers, then $\epsilon_{kij}\epsilon_{klm}=(\pm 1)(\pm 1)=1$

If $(i,j)=(m,l)$ and (i,j) are two different numbers, then $\epsilon_{kij}\epsilon_{klm}=(\pm 1)(\mp 1)=-1$

e.g., if $(i,j)=(m,l)=(2,3)$, then $\epsilon_{kij}\epsilon_{klm}=\epsilon_{123}\epsilon_{132}+\epsilon_{223}\epsilon_{232}+\epsilon_{323}\epsilon_{332}=1\times(-1)+0+0=-1$

,

If $(i, j) = (l, m)$ and (i, j) are two different numbers, then $\epsilon_{kij}\epsilon_{klm} = (\pm 1)(\pm 1) = 1$

If $(i, j) = (m, l)$ and (i, j) are two different numbers, then $\epsilon_{kij}\epsilon_{klm} = (\pm 1)(\mp 1) = -1$

Otherwise, $\epsilon_{kij}\epsilon_{klm} = 0$, because either

(1) we still have $i \neq j$ and $l \neq m$ but now (i, j, l, m) contains all three integers 1, 2, and 3, and at least one of the k indices will then equal one of the other indices for each value of $k \in (1,2,3)$, or

(2) we switch to the case in which $i = j$ or $l = m$, which would cause ϵ_{kij} or ϵ_{klm} to vanish.

Now we show that $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ has exactly the same value as $\epsilon_{kij}\epsilon_{klm}$ on the last page:

If $(i, j) = (l, m)$ and (i, j) are two different numbers, then $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 1 \times 1 - 0 \times 0 = 1$

if $(i, j) = (m, l)$ and (i, j) are two different numbers, then
$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0 \times 0 - 1 \times 1 = -1$

Otherwise, $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0$, because either

(1) we still have $i \neq j$ and $l \neq m$ but now (i, j, l, m) contains all three integers 1, 2, and 3, and there is no way to subdivide (i, j, l, m) into two pairs of matching integers, or

(2) $i = j$, in which case $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \left(\delta_{il}\delta_{im} - \delta_{im}\delta_{il}\right)_{\text{no sum over } i} = 0$, or

(3) $l = m$, in which case $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \left(\delta_{il}\delta_{jl} - \delta_{il}\delta_{jl}\right)_{\text{no sum over } l} = 0$.

Therefore, $\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

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if $(i, j) = (m, l)$ and (i, j) are two different numbers, then $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0 \times 0 - 1 \times 1 = -1$

Redo this derivation on your own as needed until you understand and memorize the identity. It's worth it because it enables you to quickly rederive many of the vector and vector-calculus identities that you will need.

Otherwise, $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0$, because either

(1) we still have $i \neq j$ and $l \neq m$ but now (i, j, l, m) contains all three integers 1, 2, and 3, and there is no way to subdivide (i, j, l, m) into two pairs of matching integers, or

(2) $i = j$, in which case $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \left(\delta_{il}\delta_{im} - \delta_{im}\delta_{il}\right)_{\text{no sum over } i} = 0$, or

(3) $l = m$, in which case $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \left(\delta_{il}\delta_{jl} - \delta_{il}\delta_{jl}\right)_{\text{no sum over } l} = 0$.

Therefore, $\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

2D Hydro

$$\nabla \times \left\{ \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \right\}$$

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$$\rightarrow \nabla \times (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} + \boldsymbol{\omega} \cancel{\nabla \cdot \boldsymbol{u}} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cancel{\nabla \cdot \boldsymbol{\omega}}$$

Vanishes because $\boldsymbol{u} = \boldsymbol{u}(x, y) \rightarrow \boldsymbol{\omega} = \omega \hat{z} \rightarrow \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = 0$

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$$\longrightarrow \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}$$

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$$\boldsymbol{\omega} \cdot \left\{ \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} \right\} \longrightarrow \frac{\partial}{\partial t} \frac{\omega^2}{2} + \boldsymbol{u} \cdot \nabla \frac{\omega^2}{2} = \nu \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega}$$

$$\longrightarrow \frac{1}{2} \frac{d}{dt} \int \mathrm{d}x\mathrm{dy}\, \omega^2 + \int \mathrm{d}x\mathrm{dy}\, \nabla \cdot \left(\boldsymbol{u} \frac{\omega^2}{2} \right) = \nu \int \mathrm{d}x\mathrm{dy}\, \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega}$$

$$\boldsymbol{\omega} \cdot \left\{ \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} \right\} \longrightarrow \frac{\partial}{\partial t} \frac{\omega^2}{2} + \boldsymbol{u} \cdot \nabla \frac{\omega^2}{2} = \nu \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega}$$

$$\longrightarrow \frac{1}{2} \frac{d}{dt} \int \mathrm{d}x\mathrm{dy}\, \omega^2 + \int \mathrm{d}x\mathrm{dy}\, \nabla \cdot \left(\boldsymbol{u} \frac{\omega^2}{2} \right) = \nu \int \mathrm{d}x\mathrm{dy}\, \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega}$$

$$\longrightarrow \frac{1}{2} \frac{d}{dt} \int \mathrm{d}x\mathrm{dy}\, \omega^2 = -\nu \int \mathrm{d}k_x \mathrm{d}k_y\, k^2 \boldsymbol{\omega}_{\boldsymbol{k}} \cdot \boldsymbol{\omega}_{-\boldsymbol{k}}$$

$$\boldsymbol{\omega}\cdot\left\{\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}=\nu\nabla^2\boldsymbol{\omega}\right\}\longrightarrow\frac{\partial}{\partial t}\frac{\omega^2}{2}+\boldsymbol{u}\cdot\nabla\frac{\omega^2}{2}=\nu\boldsymbol{\omega}\cdot\nabla^2\boldsymbol{\omega}$$

$$\longrightarrow\frac{1}{2}\frac{d}{dt}\int\mathrm{d}x\mathrm{d}y\,\omega^2+\int\mathrm{d}x\mathrm{d}y\,\nabla\cdot\left(\boldsymbol{u}\frac{\omega^2}{2}\right)=\nu\int\mathrm{d}x\mathrm{d}y\,\boldsymbol{\omega}\cdot\nabla^2\boldsymbol{\omega}$$

$$\longrightarrow\frac{1}{2}\frac{d}{dt}\int\mathrm{d}x\mathrm{d}y\,\omega^2=-\nu\int\mathrm{d}k_x\mathrm{d}k_y\,k^2\boldsymbol{\omega}_{\boldsymbol{k}}\cdot\boldsymbol{\omega}_{-\boldsymbol{k}}$$

Likewise, taking $\boldsymbol{u}\cdot\left\{\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+\nu\nabla^2\boldsymbol{u}\right\}$

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$$\longrightarrow\frac{1}{2}\frac{d}{dt}\int\mathrm{d}x\mathrm{d}y\,u^2=-\nu\int\mathrm{d}k_x\mathrm{d}k_y\,k^2\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{-\boldsymbol{k}}$$

Rates At Which Viscosity Dissipates Energy and Entropy

$$\frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,\omega^2 = -\nu \int \mathrm{d}k_x \mathrm{d}k_y\, k^2 \boldsymbol{\omega}_{\boldsymbol{k}} \cdot \boldsymbol{\omega}_{-\boldsymbol{k}}$$

Use $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$

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How can we prove this identity?

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = \epsilon_{ijk}A_jB_k\epsilon_{ilm}C_lD_m$$

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$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m$$

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$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m = (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$$

Rates At Which Viscosity Dissipates Energy and Entropy

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$$\boldsymbol{k}\cdot\boldsymbol{u}_k = 0 \quad\longrightarrow\quad \boldsymbol{\omega}_k\cdot\boldsymbol{\omega}_{-k} = (i\boldsymbol{k}\times\boldsymbol{u}_k)\cdot(-i\boldsymbol{k}\times\boldsymbol{u}_{-k}) = k^2\boldsymbol{u}_k\cdot\boldsymbol{u}_{-k}$$

Rates At Which Viscosity Dissipates Energy and Entropy

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$$\frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,\omega^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^4\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{-\boldsymbol{k}}$$

Rates At Which Viscosity Dissipates Energy and Entropy

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$$\frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,u^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^2\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{-\boldsymbol{k}}$$

Why Energy Undergoes an Inverse Cascade in 2D Hydro

$$\frac{\mathrm{d}W}{\mathrm{d}t} \equiv \frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,\omega^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^4\boldsymbol{u_k}\cdot\boldsymbol{u_{-k}}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} \equiv \frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,u^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^2\boldsymbol{u_k}\cdot\boldsymbol{u_{-k}}$$

Suppose forcing injects energy and enstrophy into the turbulence at a finite rate at the outer scale L and that energy cascades at this same rate to very small scales where it viscously dissipates. Now keep reducing ν. In 3D hydro, this works fine: all that happens is that the inertial-range energy spectrum keeps extending to higher and higher k. But in 2D hydro, the rate of enstrophy destruction would diverge in this scenario, because the integrand in $\mathrm{d}W/\mathrm{d}t$ above contains a higher power of k than the integrand in $\mathrm{d}E/\mathrm{d}t$.

$\longrightarrow$ contradiction. $\longrightarrow$ Energy cannot cascade to smaller scales.

Twin Cascades in 2D Hydro (Fjørtøft 1953)

$$\frac{\mathrm{d}W}{\mathrm{d}t} \equiv \frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,\omega^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^4\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{-\boldsymbol{k}}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} \equiv \frac{1}{2}\frac{d}{dt}\int \mathrm{d}x\mathrm{d}y\,u^2 = -\nu\int \mathrm{d}k_x\mathrm{d}k_y\,k^2\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{-\boldsymbol{k}}$$

Because nonlinear interactions cannot transfer energy to smaller scales, they transfer energy to larger scales. This is called an inverse cascade.

On the other hand, the enstrophy does cascade to smaller scales, where it dissipates viscously.

Because the energy undergoes an inverse cascade, it undergoes almost no viscous dissipation, and the turbulence is very long lived.

Outline

1. Intro: what is turbulence?
2. Review: continuity equation, Navier-Stokes equation
3. Review: Fourier transforms
4. Energy cascade from large scales to small scales in 3D hydro
5. Inverse energy cascade from small scales to large scales in 2D hydro
6. Extras: turbulent transport, turbulent heating, passive-scalar diffusion

Turbulent Transport

Suppose particles undergo a random walk, taking random steps of length Δx during each time Δt. Their density then satisfies the diffusion equation $\frac{\partial n}{\partial t} = D\nabla^2 n$, where $D \sim (\Delta x)^2/\Delta t$. (We saw earlier how to solve this equation).

In a turbulent fluid (or plasma), the dominant diffusion process is often from the turbulent motions of fluid elements, which take steps of length the outer scale L once every outer-scale eddy turnover time L/u_L, leading to the diffusion coefficient $D_{\rm turb} \sim \frac{L^2}{L/u_L} = Lu_L$

This can lead to, for example, diffusion of angular momentum in accretion disks, cosmic rays in the interstellar medium, thermal energy in convective stars.

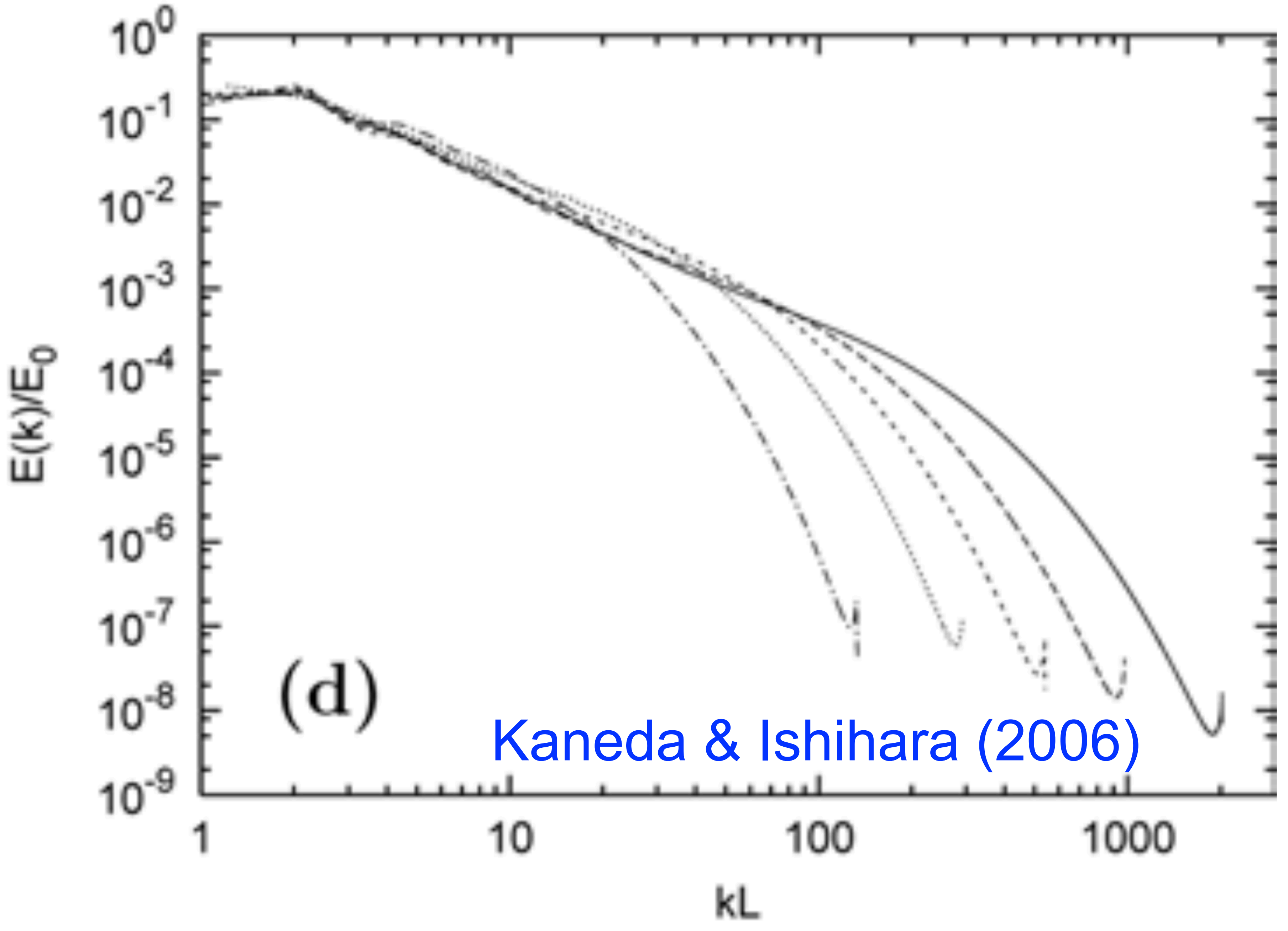
Turbulent Heating

The energy that is dissipated at small scales heats the ambient medium. This heating can be important in the solar wind and galaxy-cluster plasmas.

Passive Scalar Diffusion

Cream in coffee, pollutants in the atmosphere, are advected by the background fluid. If the background fluid is turbulent, the “passive scalar” density fluctuations (e.g., in the density of the cream, or the concentration of pollutants) pick up a power law spectrum. In hydro turbulence, the spectral index of this spectrum within the inertial range has the same Kolmogorov scaling as the velocity fluctuations.

Conclusion



- In 3D hydro, energy (1) is injected or initialized at small k, (2) cascades to larger k through the inertial range, and (3) dissipates at large k.

- In the inertial range, the energy flux $\epsilon \sim u_\lambda^3/\lambda$ is independent of $\lambda \sim 1/k$, which yields the Kolmogorov scaling $u_\lambda \propto \lambda^{1/3}$, which is equivalent to $E(k) \propto k^{-5/3}$.

- As $\nu \to 0$, the energy cascade rate and dissipation rate $\to u_{\rm rms}^3/L$, independent of ν

- In 2D hydro, energy undergoes an inverse cascade to larger scales, while enstrophy cascades to smaller scales.